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Degree Reduction of Interval B-Spline Curves*

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Abstract: Two different algorithmic approaches, the integral method and the piecewise degree reduction method, to the degree reduction of interval B-spline curves are presented. Examples are provided to demonstrate the algorithms and to compare the two different approaches. The experimental results show that the piecewise degree reduction method is generally more efficient and produces tighter bound than the integral method.

Key words: interval B-spline curve; interval arithmetic; degree reduction; knot removal

In Computer Aided Design and Geometric Modeling, there are considerable interests in approximating curves and surfaces with simpler forms of curves and surfaces. This problem arises whenever CAD data need to be shared across heterogeneous systems which use different proprietary data structures for model representations. For example, some systems restrict themselves to polynomial forms or limit the polynomial degree that they accommodate.

In the past decade, a lot of research work has been focused on the problem of degree reduction of B-spline curves[1~4]. However, the research is concerned with how good the approximation is; none of it deals with the numerical gaps. Such gaps, though extremely small in size, could cause geometric modeling system's failure, generate useless analysis results, and create defects in finished products. Models with gaps often need tremendous rework at the receiving end of data exchange[5].

To overcome the above difficulty, Sederberg *et al.* introduced interval representation forms of curves and surfaces that can embody a complete description of coefficient errors[6]. Recently, Hu *et al.* and Tuohy *et al.*[7~11] turned to interval forms of geometric objects and rounded interval arithmetic to deal with the problems of valid and consistent representations of geometric objects and robust geometric operations such as curve/curve and surface/surface intersections. Their results indicate that using interval arithmetic will substantially increase the numerical stability in geometric computations, and thus enhance the robustness of current CAD/CAM systems.

In a previous work, we dealt with the problem of degree reduction of interval Bézier curves, i.e., bounding an interval Bézier curve with a lower degree interval Bézier curve.[12] Our result is significantly superior to Rokne's result.[13] In this paper, we will extend the idea to interval B-spline curves, that is, we will develop algorithms to

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bound an interval B-spline curve with a lower degree interval B-spline curve. Although the main algorithms presented in this paper share some similarities with those for the degree reduction of interval Bézier curves, it is still valuable to fully describe the algorithms due to some special properties of interval B-spline curves.

The organization of the paper is as follows. In the next section, some fundamental concepts about interval arithmetic and interval B-spline curves are introduced. Then in Section 2, two different algorithms are presented to bound an interval B-spline curve with an interval B-spline curve of one degree lower. Finally in Section 3, we provide some examples to demonstrate the algorithms and make some comparisons between the two different algorithms.

1 Interval B-Spline Curves

1.1 Interval arithmetic

An interval $[a,b]$ is the set of real numbers:

$$[a,b]=\{x \mid a\le x\le b\}$$

where the following interval arithmetic operations are defined:

$$\begin{aligned}
&[a,b]+[c,d]=[a+c,b+d]\\
&[a,b]-[c,d]=[a-d,b-c]\\
&[a,b]\times[c,d]=[\min(ac,ad,bc,bd),\max(ac,ad,bc,bd)]\\
&[a,b]/[c,d]=[a,b]\times[1/d,1/c],\quad 0\notin[c,d]
\end{aligned}\tag{1}$$

It is easy to verify that interval addition and multiplication are commutative and associative, but that multiplication does not, in general, distribute over addition.

Interval analysis first emerged as a tool in numerical mathematics to enable digital computers to execute algorithms capturing all the round off errors automatically. It offers an essentially infallible way to monitor error propagation in numerical algorithms that use floating-point arithmetic. Many familiar algorithms can be reformulated to accept interval operands. For applications of interval analysis in other areas, the reader is referred to Moore's book *Interval Analysis*[14,15].

1.2 Interval B-spline curve

An interval B-spline function is a B-spline function whose coefficients are intervals:

$$[B](u):=\sum_{i=0}^{n}[a_i,b_i]N_i^p(u),\tag{2}$$

where $N_i^p(u)$ is the B-spline basis function of order p associated with knot sequence $U:=\{u_0,u_1,\ldots,u_{n+p}\}$, where $u_0\le u_1\le\ldots\le u_{n+p}$ and $u_i<u_{i+p}$, for i between 0 and n. When $u_i=u_0+h_i,\ i=1,2,\ldots,n+p$, $N_i^p(u)$ is called a uniform B-spline. Otherwise it is called a non-uniform B-spline.

Since the B-spline basis functions $N_i^p(u)$ are all nonnegative over interval $u\in(-\infty,\infty)$, interval B-spline function (2) can also be rewritten as:

$$[B](u):=[B_{\min}(u),B_{\max}(u)]\tag{3}$$

where

$$B_{\min}(u)=\sum_{i=0}^{n}a_iN_i^p(u),\quad B_{\max}(u)=\sum_{i=0}^{n}b_iN_i^p(u),\tag{4}$$

$B_{\min}(u)$ and $B_{\max}(u)$ are called lower bound (denoted by $lb([B](u))$) and upper bound (denoted by $ub([B](u))$) of interval B-spline function (2) respectively.

The width of an interval B-spline function is defined by

$$w([B](u)) = \left\| B_{\max}(u) - B_{\min}(u) \right\|. \tag{5}$$

here the norm $\|\cdot\|$ is the standard norm such as $\|\cdot\|_1, \|\cdot\|_2$.

An interval B-spline curve is a B-spline curve whose control points are vector-valued intervals (i.e. rectangular regions):

$$[\boldsymbol{B}](u) = \sum_{i=0}^{n} [\boldsymbol{P}_i] N_i^p(u), \tag{6}$$

where $\boldsymbol{P}_i = [a_i, b_i] \times [c_i, d_i]$, $i = 0,1,...,n$.

The interval B-spline curve (6) can also be written in vector form:

$$[\boldsymbol{B}](u) = ([x])(u), [y](u)), \tag{7}$$

where

$$[x](u) = \sum_{i=0}^{n} [a_i, b_i] N_i^p(u), \quad [y](u) = \sum_{i=0}^{n} [c_i, d_i] N_i^p(u), \tag{8}$$

are interval B-spline functions. The width of an interval B-spline curve is defined to be the maximum of the widths of the $[x](u)$ and $[y](u)$.

Interval B-spline curves share many properties with ordinary B-spline curves. For details, we refer the reader to Ref.[11].

2 Degree Reduction of Interval B-Spline Curves

We begin with the problem of degree reduction of interval B-spline functions. Two algorithms are developed to solve the problem. The first algorithm, the so called *integral method*, provides an integral way to reduce the degree of an interval B-spline function, while the second algorithm, the so called *piecewise degree reduction method*, extracts each segment of an interval B-spline function and then reduces the degree of the interval polynomial for each segment.

Unlike the degree reduction of interval polynomials, the problem of degree reduction of interval B-spline functions is different for uniform interval B-spline functions and non-uniform interval B-spline functions. For the uniform case, the problem can be stated as follows.

Problem 1. Given a uniform interval B-spline function of order p with knot vector

$U = \{u_0, u_1, ..., u_{n+p}\}$, and $u_i = u_0 + ih$, $i = 0,1,...,n+p$:

$$[P](u) = \sum_{i=0}^{n} [p_i] N_i^p(u) \tag{9}$$

find another uniform interval B-spline function of order p-1 with knot vector $V = \{u_1, ..., u_{n+p-1}\}$:

$$[Q](u) = \sum_{i=1}^{n} [q_i] N_i^{p-1}(u) \tag{10}$$

such that

$$[P](u) \subset [Q](u), \quad u \in [u_{p-1}, u_{n+1}] \tag{11}$$

and the width of $[Q](u)$ is as small as possible.

For the non-uniform case, the problem can be described:

Problem 2. Given a non-uniform interval B-spline function of order p:

$$[P](u) = \sum_{i=0}^{n} [p_i] N_i^p(u), \tag{12}$$

where the knot vector U takes the form

$$U=\{\underbrace{u_0,\ldots,u_0}_{p},\underbrace{u_1,\ldots,u_1}_{m_1},\underbrace{u_2,\ldots,u_2}_{m_2},\ldots,\underbrace{u_s,\ldots,u_s}_{m_s},\underbrace{u_{s+1},\ldots,u_{s+1}}_{p}\}$$

$m_i<p$, $i=1,2,\ldots,s$, find another non-uniform interval B-spline function of order $p-1$:

$$[Q](u)=\sum_{i=0}^{\hat{n}}[q_i]N_i^{p-1}(u) \tag{13}$$

with knot vector

$$V=\{\underbrace{u_0,\ldots,u_0}_{p-1},\underbrace{u_1,\ldots,u_1}_{\hat{m}_1},\underbrace{u_2,\ldots,u_2}_{\hat{m}_2},\ldots,\underbrace{u_s,\ldots,u_s}_{\hat{m}_s},\underbrace{u_{s+1},\ldots,u_{s+1}}_{p-1}\},$$

where $\hat{n}=p-2+\sum_{i=1}^{s}\hat{m}_i$, and $\hat{m}_i=1$ when $m_i=1$; $\hat{m}_i=m_i-1$ when $m_i>1$, $i=1,2,\ldots,s$ such that

$$[P](u)\subset[Q](u),\quad u\in[u_0,u_{s+1}] \tag{14}$$

and the width of $[Q](u)$ is as small as possible.

To solve the above problems, we can first solve the following

Problem 3. Given a uniform B-spline function of order p with knot vector: $U=\{u_0,u_1,\ldots,u_{n+p}\}$:

$$f(u)=\sum_{i=0}^{n}f_iN_i^p(u), \tag{15}$$

find another uniform B-spline function of order $p-1$ with knot vector $V=\{u_1,\ldots,u_{n+p-1}\}$:

$$g(u)=\sum_{i=0}^{n}g_iN_i^{p-1}(u), \tag{16}$$

such that
$$g(u)\ge f(u),\quad u\in[u_{p-1},u_{n+1}] \tag{17}$$
and
$$\|f-g\| \tag{18}$$
is minimized.

Problem 4. Given a non-uniform B-spline function of order p:

$$f(u)=\sum_{i=0}^{n}f_iN_i^p(u) \tag{19}$$

with the knot vector

$$U=\{\underbrace{u_0,\ldots,u_0}_{p},\underbrace{u_1,\ldots,u_1}_{m_1},\underbrace{u_2,\ldots,u_2}_{m_2},\ldots,\underbrace{u_s,\ldots,u_s}_{m_s},\underbrace{u_{s+1},\ldots,u_{s+1}}_{p}\}$$

find another non-uniform B-spline function of order $p-1$:

$$g(u)=\sum_{i=0}^{\hat{n}}g_iN_i^{p-1}(u) \tag{20}$$

with knot vector

$$V=\{\underbrace{u_0,\ldots,u_0}_{p-1},\underbrace{u_1,\ldots,u_1}_{\hat{m}_1},\underbrace{u_2,\ldots,u_2}_{\hat{m}_2},\ldots,\underbrace{u_s,\ldots,u_s}_{\hat{m}_s},\underbrace{u_{s+1},\ldots,u_{s+1}}_{p-1}\}$$

where $\hat{n}=p-2+\sum_{i=1}^{s}\hat{m}_i$, and $\hat{m}_i=1$ when $m_i=1$; $\hat{m}_i=m_i-1$ when $m_i>1$, $i=1,2,\ldots,s$, such that

$$g(u)\ge f(u),\quad u\in[u_0,u_{s+1}] \tag{21}$$

and

$$\|f-g\| \tag{22}$$

is minimized.

In the following, we will develop two different methods to solve the above problems.

2.1 Integral method

In this subsection, we will develop an integral method to solve Problem 3 and Problem 4 which is based on perturbations to the coefficients of the original B-spline function. We define

$$\|f-g\|:=\int_{u_{p-1}}^{u_{n+1}}(f(u)-g(u))\mathrm{d}u\ .$$

Lemma 1.[3] Given a uniform B-spline function of order p with knot vector $U=\{u_0,u_1,...,u_{n+p}\}$:

$$B(u)=\sum_{i=0}^{n}d_iN_i^p(u)$$

then $B(u)$ can be exactly represented as a uniform B-spline of order p-1 if and only if

$$\sum_{j=i-p+1}^{i}(-1)^{i-j}d_j\binom{p-1}{i-j}=0\ ,\quad i=p-1,p,...,n \tag{23}$$

Furthermore, when condition (23) holds, $B(u)$ can be represented as a uniform B-spline of order p-1 with knot vector $\overline{U}=\{u_1,...,u_{n+p-1}\}$:

$$\overline{B}(u)=\sum_{i=1}^{n-1}\overline{d}_iN_i^{p-1}(u)$$

where
$$\begin{cases}[\overline{d}_1,\ldots,\overline{d}_{p-1}]^T=\dfrac{1}{h(p-1)}A^{-1}B\\ \overline{d}_{i+1}=\dfrac{1}{h(p-1)}\displaystyle\sum_{j=i-p+2}^{i+1}d_j\sum_{l=0}^{i-j+1}(-1)^{l+1}(p-1)\binom{p}{l}u_{j+l}-\sum_{j=i-p+3}^{i}(-1)^{i-j+1}\overline{d}_j\binom{p-2}{i-j+1}\\ i=p-1,p,...,n-1\end{cases}$$

with $h=u_{i+1}-u_i$, $A=(a_{ij})$ is a square matrix of order p-1 with elements

$$(a_{ij})=\sum_{l=0}^{p-j-1}(-1)^l\binom{p-1}{l}\binom{p-2}{p-1-i}(-u_{j+l})^{i-1}\ ,\quad i,j=1,2,...,p-1$$

and $B=[b_1,\ldots,b_{p-1}]^T$ is a column vector with elements

$$b_i=\sum_{j=0}^{p-1}\sum_{l=0}^{p-j-1}(-1)^l\binom{p}{l}\binom{p-1}{p-1-i}(-u_{j+l})^i\ ,\quad i=1,2,...,p-1$$

The above lemma provides a way to solve Problem 3. The idea is as follows. We perturb each of the coefficients of B-spline function $f(u)$ with some nonnegative value ε_i such that the perturbed B-spline function $g(u):=\sum_{i=0}^{n}(f_i+\varepsilon_i)N_i^p(u)$ satisfies the degeneracy condition (23), then $g(u)$ is an upper bound of $f(u)$. Since

$$\|f-g\|=\int_{u_{p-1}}^{u_{n+1}}\varepsilon_iN_i^p(u)=h\sum_{i=0}^{n}\varepsilon_i.$$

Problem 3 can be converted into the following linear programming problem:

$$\begin{cases}\text{Min}\ \ \sum_{i=0}^{n}\varepsilon_i\\ \text{s.t.}\ \ \sum_{j=i-p+1}^{i}(-1)^{i-j}(f_i+\varepsilon_i)\binom{p-1}{i-j}=0,\ i=p-1,p,...,n\\ \varepsilon_i\geq 0,\ i=0,1,...,n\end{cases} \tag{24}$$

To further reduce the size of the above linear programming problem, we first solve for ε_i, i=0,1,…,p–1from the degeneracy condition:

$$\varepsilon_i = \phi_i(\varepsilon_p,\ldots,\varepsilon_n),\quad i = 0,1,\ldots,p-1$$

and then substitute the above equations into (24). Thus we only have to solve

$$\begin{cases} \text{Min}\ \ \Phi(\varepsilon_p,\ldots,\varepsilon_n) \\ \text{s.t.}\ \ \phi_i(\varepsilon_p,\ldots,\varepsilon_n) \ge 0,\ i = 0,1,\ldots,p-1 \\ \varepsilon_i \ge 0,\ i = p,\ldots,n \end{cases} \tag{25}$$

where $\Phi = \sum_{i=0}^{p-1}\phi_i(\varepsilon_p,\ldots,\varepsilon_n) + \sum_{i=p}^{n}\varepsilon_i$. After solving the above linear programming problem, we find a solution for Problem 3.

For the non-uniform B-spline case, the integral method also works, since the degeneracy condition for non-uniform B-spline functions also exists (see Ref.[4] for a reference). But it will be much more complicated. We will not go into details but only illustrate an example in the last section.

2.2 Piecewise degree reduction method

In this subsection, we present a piecewise degree reduction approach to solve Problem 3 and Problem 4. The main procedure of the algorithm is as follows. We first subdivide the B-spline function into a series of Bézier segments, then reduce the degree of each Bézier segment by the methods proposed in Ref.[12]. Finally we combine the degree reduced Bézier segments into a single B-spline which is modified to the required smoothness. We will take non-uniform B-spline case as an example to demonstrate the algorithm.

The first step of the algorithm is to extract the Bézier segments from the given B-spline function $f(u)$. To do this, we insert knots at each inner knot position t_i for $p-m_i-1$ times, $i=1,2,\ldots,m_s$, then the B-spline function is automatically subdivided into $s+1$ Bézier segments. Assume the knot vector takes the form

$$U = \{\underbrace{u_0,\ldots,u_0}_{p}, u_p,\ldots,u_n, \underbrace{u_{s+1},\ldots,u_{s+1}}_{p}\}, \tag{26}$$

where $n = \sum_{i=1}^{s} m_i + p - 1$. The insertion formula for inserting a knot r times is as follows[1,16]:

$$\begin{cases} d_i^r = \alpha_i^r d_i^{r-1} + (1-\alpha_i^r) d_{i-1}^{r-1} \\ \alpha_i^r = \dfrac{u-u_i}{u_{i+k-r+1}-u_i} \\ d_i^0 = d_i, k-p+r+1 \le i \le k-w \end{cases} \tag{27}$$

where $u \in [u_k, u_{k+1})$ is the knot to be inserted, and w is the multiplicity of u_k.

After extracting the Bézier segments (polynomials) of the B-spline function, we find an upper(lower) polynomial bound for each Bézier polynomial using the techniques developed in Ref.[12].

Our last step is to combine the upper(or lower) bound Bézier segments to form a B-spline function of order $p-1$. However, the B-spline function obtained by this way may not have required continuity at the knots, or equivalently the multiplicities of the knots do not coincide with the requirements (generally they are greater than requirements). Thus we have to remove each inner knot, of course approximately, for certain number of times. Eck *et al*. provided a nice algorithm for knot removal[17], and they also gave an error bound between the orginal B-spline curve and the knot-removed B-spline curve. We refer the reader to Ref.[17] for the details.

Notice that, in general, knot removal will produce an approximation but not necessarily an upper bound of the original B-spline function. To make the knot-removed B-spline an upper bound, we simply add the maximum error into it.

2.3 Degree reduction of interval B-spline curve

After the problem of degree reduction of interval B-spline functions has been solved, the problem of degree reduction for interval B-spline curves can be solved easily. The basic idea is based on the following

Proposition 1. Let

$$[\boldsymbol{B}](u)=([x](u),[y](u))$$

be a B-spline curve of order p, where

$$[x](u)=\sum_{i=0}^{n}[a_i,b_i]N_i^p(u),\quad [y](u)=\sum_{i=0}^{n}[c_i,d_i]N_i^p(u).$$

Suppose

$$[\bar{x}](u)=\sum_{i=0}^{n-1}[\bar{a}_i,\bar{b}_i]N_i^{p-1}(u),\quad [\bar{y}](u)=\sum_{i=0}^{n-1}[\bar{c}_i,\bar{d}_i]N_i^{p-1}(u)$$

are the degree reduced interval B-splines of $x(u)$ and $y(u)$ respectively, then

$$[\bar{\boldsymbol{B}}](u)=([\bar{x}](u),[\bar{y}](u))$$

is a B-spline curve of order $p-1$ which bounds $[\boldsymbol{B}](u)$.

Proof. Straightforward.

3 Examples and Comparisons

Now we will provide two examples to demonstrate the algorithms for reducing the degree of an interval B-spline curve. We will also make some comparisons between the two different approaches.

To measure how tight the degree reduced curve bounds the original curve, we define the bounding ratio by

$$e:=\frac{\max(\max_{i=0}^{n-1}(\bar{d}_i-\bar{c}_i),\max_{i=0}^{n-1}(\bar{b}_i-\bar{a}_i))}{\max(\max_{i=0}^{n}(d_i-c_i),\max_{i=0}^{n}(b_i-a_i))},\tag{28}$$

where $[\bar{a}_i,\bar{b}_i]\times[\bar{c}_i,\bar{d}_i]$, i=0,1,…,n-1 are the control points of degree reduced curve, and $[a_i,b_i]\times[c_i,d_i]$, i=0,1,…,n are the control points of the original B-spline curve. Of course, the smaller the bounding ratio is, the tighter the bound is.

Example 1. Given a uniform interval B-spline curve of order 5 whose knot vector and the control points are as follows:

$$U=\{0,1,2,3,4,5,6,7,8,9,10,11,12,13\}$$

$$\begin{cases}[d_0]=[210,230]\times[25,50], & [d_1]=[10,22]\times[100,120]\\ [d_2]=[0,15]\times[230,250], & [d_3]=[120,130]\times[235,300]\\ [d_4]=[320,335]\times[290,320], & [d_5]=[510,530]\times[300,320]\\ [d_6]=[650,680]\times[220,260], & [d_7]=[630,650]\times[102,130]\\ [d_8]=[430,460]\times[-65,-50]\end{cases}$$

By integral method, the problem of reducing the degree of the given B-spline curve can be converted into solving a linear programming problem with 4 variables. After solving the linear programming problem, the control points of the reduced B-spline curve are obtained as:

$$\begin{cases}[\bar{d}_1]=[87.16,109.13]\times[62.06,106.36], & [\bar{d}_2]=[-14.84,4.08]\times[137.62,206.98]\\ [\bar{d}_3]=[49.16,72.08]\times[206.14,283.86], & [\bar{d}_4]=[213.16,243.96]\times[255.37,329.48]\\ [\bar{d}_5]=[378.16,430.08]\times[301.24,345.60], & [\bar{d}_6]=[565.43,599.32]\times[240.23,301.98]\\ [\bar{d}_7]=[630.12,671.31]\times[180.32,220.12], & [\bar{d}_8]=[530.34,563.35]\times[13.12,72.46]\end{cases}$$

The maximum perturbation and the bounding ratio are 55.9 and 1.2, respectively. Figure 1 illustrates the corresponding interval B-spline curves.

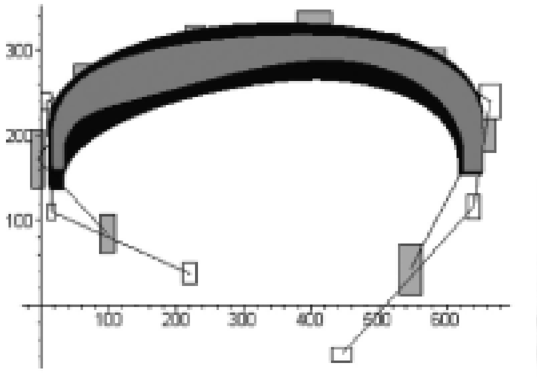

Fig.1 Reducing the degree of a uniform interval B-spline by integral method

To reduce the degree of the original B-spline curve by piecewise degree reduction method, we follow the steps:

Insert each inner knot (5,6,7,8) three times, and five interval Bézier pieces are obtained.

Reduce the degree of each of the three interval Bézier curves by the techniques in Ref.[12].

Two occurrences of the inner knots are removed. Notice how knot removal introduces an additional error that affects more than one-knot span.

Compute the error by knot removal , and add the error to the knot-removed curve.

The control points of the new interval B-spline curve are

$$\begin{cases} [\bar{d}_1]=[201.47,238.23]\times[17.60,56.31], & [\bar{d}_2]=[-33.13,-13.38]\times[124.78,157.78] \\ [\bar{d}_3]=[44.27,79.75]\times[239.95,296.23], & [\bar{d}_4]=[218.73,251.38]\times[260.75,326.52] \\ [\bar{d}_5]=[382.23,420.44]\times[294.57,327.66], & [\bar{d}_6]=[590.55,620.37]\times[254.42,313.28] \\ [\bar{d}_7]=[660.23,692.54]\times[158.32,204.21], & [\bar{d}_8]=[424.19,468.22]\times[-67.53,-44.32] \end{cases}$$

The bounding ratio by this method is 1.0. Figure 2 shows the original and the degree reduced interval B-spline curves.

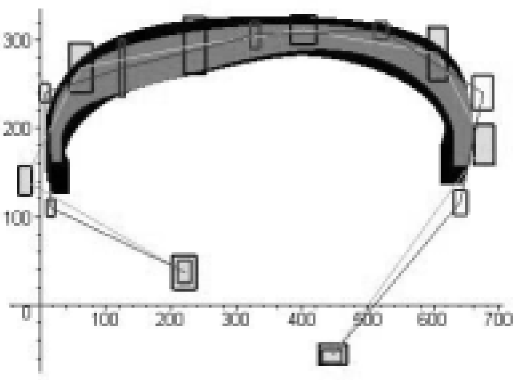

Fig.2 Reducing the degree of a uniform interval B-spline by piecewise degree reduction method

Example 2. Let $[\boldsymbol{B}](u)$ be a non-uniform B-spline curve of order 5 with knot vector:

$$U=\{0,0,0,0,0,0,1.5,1.5,3,4.5,4.5,6,6,6,6,6,6\}$$

and the control points:

$$\begin{cases} [d_0]=[-1.2,-1.15]\times[-0.7,-0.61], & [d_1]=[-0.5,-0.43]\times[-0.5,-0.44] \\ [d_2]=[-0.5,-0.38]\times[0.3,0.38], & [d_3]=[-1.0,-0.91]\times[0.2,0.32] \\ [d_4]=[-0.9,-0.85]\times[0.9,1.01], & [d_5]=[-0.1,-0.02]\times[1.2,1.32] \\ [d_6]=[0.7,0.78]\times[0.7,0.78]\ , & [d_7]=[0.7,0.78]\times[0.2,0.26] \\ [d_8]=[0.3,0.35]\times[0.3,0.35], & [d_9]=[0.3,0.37]\times[-0.5,-0.46] \\ [d_{10}]=[0.9,1.0]\times[-0.7,-0.62] & \end{cases}$$

The degree reduced interval curve by the integral method is a degree three-interval B-spline curve with the knot vector U={0,0,0,0,0.3,0.6,1,1,1,1} and the control points

$$\begin{cases} [\bar{d}_0]=[-1.19,-1.08]\times[-0.71,-0.60], & [\bar{d}_1]=[-0.31,-0.19]\times[-0.16,-0.04] \\ [\bar{d}_2]=[-1.14,-0.96]\times[0.40,0.54], & [\bar{d}_3]=[-0.90,-0.79]\times[1.02,1.20] \\ [\bar{d}_4]=[0.50,0.64]\times[0.96,1.16], & [\bar{d}_5]=[0.79,0.95]\times[0.38,0.49] \\ [\bar{d}_6]=[0.10,0.21]\times[-0.20,-0.10], & [\bar{d}_7]=[0.87,1.01]\times[-0.71,-0.61] \end{cases}$$

The bound ratio is e=5.36. Figure 3 shows the results.

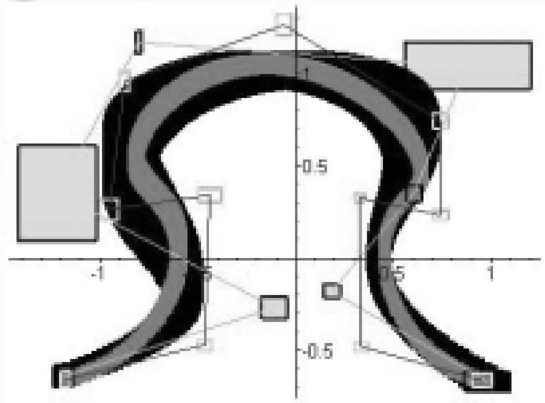

Fig.3 Reducing the degree of a non-uniform B-spline curve by integral method

By the piecewise degree reduction method, the control points of the resulting interval curve are

$$\begin{cases} [\bar{d}_1]=[-1.20,-1.08]\times[-0.71,-0.60], & [\bar{d}_2]=[-0.31,-0.19]\times[-0.16,-0.04] \\ [\bar{d}_3]=[-1.14,-0.96]\times[0.40,0.54], & [\bar{d}_4]=[-0.90,-0.79]\times[1.02,1.20] \\ [\bar{d}_5]=[0.50,0.64]\times[0.96,1.16], & [\bar{d}_6]=[0.79,0.95]\times[0.38,0.49] \\ [\bar{d}_7]=[0.10,0.21]\times[-0.20,-0.10], & [\bar{d}_8]=[0.87,1.01]\times[-0.71,-0.61] \end{cases}$$

The bounding ratio is e=1.67. Figure 4 depicts the original and degree reduced interval curves. Note that in this example, the inner knots 0.3 and 0.6 are inserted twice and removed twice.

From tens of examples we have tested so far, we conclude that the integral method generally produces looser bound than the piecewise degree reduction method both for uniform interval B-spline curves and non-uniform interval B-spline curves. Furthermore, piecewise degree reduction method is generally faster than integral method since a linear programming problem need be solved for integral method, while only some knots insertion and knots removal are needed for piecewise degree reduction method.

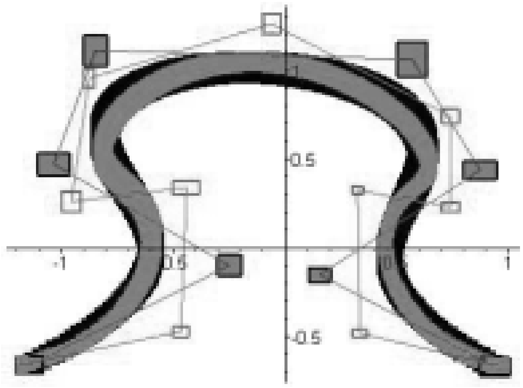

Fig.4 Reducing the degree of a non-uniform B-spline curve by piecewise degree reduction method

4 Conclusion

In this paper, we developed two methods——the integral method and the piecewise degree reduction method to solve the problem of the degree reduction of interval B-spline curves. The piecewise degree reduction method is generally more efficient than integral method. Furthermore, it produces tighter bound than integral method. The idea in this paper can be generalized to solve the problem of degree reduction of interval B-spline surfaces. We will discuss it in another paper.

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区间 B 样条曲线的降阶

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摘要: 提出了区间 B 样条曲线的降阶问题,并给出了两种不同的降阶方法:整体降阶法与分段降阶法.给出了两个实例来说明算法,并比较了两种算法的优劣.实验结果表明,无论对均匀区间 B 样条曲线还是非均匀区间 B 样条曲线,分段降阶方法比整体降阶方法都更有效,而且能得到更紧的降阶区间 B 样条曲线.

关键词: 区间 B 样条曲线;区间运算;降阶;移去节点

中图法分类号: TP391 文献标识码: A

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