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Branching Elements in $\mathbf{R}_{wtt}/\mathbf{M}_{wtt}$*

ZHANG Zai-yue SUI Yue-fei

(Department of Computer Science Industry College Yangzhou University Yangzhou 225009)

(Institute of Software The Chinese Academy of Sciences Beijing 100080)

E-mail: yzzzy@public.yz.jsinfo.cn/syf@ox.ios.ac.cn

Abstract It is proved that every element $[\mathbf{c}]\in \mathbf{R}_{wtt}/\mathbf{M}_{wtt}$ except the greatest and least elements is branching in $\mathbf{R}_{wtt}/\mathbf{M}_{wtt}$, i.e., the greatest lower bound of some two elements greater than $[\mathbf{c}]$, where $\mathbf{R}_{wtt}/\mathbf{M}_{wtt}$ is the quotient of the r.e. wtt-degrees $\mathbf{R}_{wtt}$ modulo the cappable r.e. wtt-degrees $\mathbf{M}_{wtt}$.

Key words Recursively enumerable degree, weak truth table reduction.

1 Introduction

Let $\mathbf{R}$ be the upper semilattice of the r.e. degrees, $\mathbf{M}$ the set of all the cappable r.e. degrees, and $\mathbf{NC}$ the set of all the noncappable r.e. degrees. Ambos-Spies, Jockusch, Shore and Soare[1] proved that $\mathbf{M}$ is an ideal in $\mathbf{R}$. Hence, we have a new structure $\mathbf{R}/\mathbf{M}$, the quotient of the r.e. degrees $\mathbf{R}$ modulo the cappable r.e. degrees $\mathbf{M}$ (we shall use $[\mathbf{a}],[\mathbf{b}],\ldots$ to denote the elements of $\mathbf{R}/\mathbf{M}$). Schwarz[2] proved that $\mathbf{R}/\mathbf{M}$ is downward dense. Ambos-Spies (quoted in Ref. [3]) commented that the downward density theorem in $\mathbf{R}/\mathbf{M}$ follows from the Robinson's splitting theorem and the fact that $\mathbf{NC}=\mathbf{LCu}$, the set of all the r.e. degrees which cup to $\mathbf{0}'$ by low r.e. degrees. Sui and Zhang[4] proved that the Shoenfield cupping conjecture holds in $\mathbf{R}/\mathbf{M}$, i.e., given any $[\mathbf{a}],[\mathbf{b}]\in\mathbf{R}/\mathbf{M}$ such that $[\mathbf{0}]<[\mathbf{a}]<[\mathbf{b}]$ there exists an r.e. degree $\mathbf{c}$ such that $[\mathbf{c}]<[\mathbf{b}]$ and $[\mathbf{b}]=[\mathbf{a}]\vee[\mathbf{c}]$. Sui[5] and Yi[6], independently, proved that the Shoenfield conjecture does not hold in $\mathbf{R}/\mathbf{M}$. But we do not know whether there is a branching element in $\mathbf{R}/\mathbf{M}$.

In this paper we shall consider the quotient $\mathbf{R}_{wtt}/\mathbf{M}_{wtt}$. In the following sections, we assume that every degree mentioned is an r.e. wtt-degree. We shall prove that for every $[\mathbf{c}]\in\mathbf{R}_{wtt}/\mathbf{M}_{wtt}$ such that $[\mathbf{0}]<[\mathbf{c}]<[\mathbf{0}']$ there are $[\mathbf{a}]$ and $[\mathbf{b}]\in\mathbf{R}_{wtt}/\mathbf{M}_{wtt}$ such that $[\mathbf{c}]<[\mathbf{a}],[\mathbf{b}]$ and $[\mathbf{c}]=[\mathbf{a}]\wedge[\mathbf{b}]$.

Our notation is standard, as described by Soare[7] with a minor change. A number x is *unused* at stage $s+1$ if $x\geq s$ is greater than any number used so far in the construction. We use $\Phi,\Psi,\Theta,\ldots$ to denote weak truth table functionals, as usual, which have recursive use functions $\varphi,\psi,\theta,\ldots$, respectively, increasing in argument. Before ending this section, we list some basic definitions and theorems that are used in the following sections.

Definition 1.1. (i) An r.e. degree $\mathbf{a}$ is *cappable* if there exists an r.e. degree $\mathbf{b}>\mathbf{0}$ such that $\mathbf{a}\cap\mathbf{b}=\mathbf{0}$; and

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ZHANG Zai-yue was born in 1961. He received the Ph.D. degree in mathematics from the Institute of Software, the Chinese Academy of Sciences in 1995. He is an associate professor of the Department of Computer Science, Industry College, Yangzhou University. His current research interests include recursion theory, complexity of computation and set theory. **SUI Yue-fei** was born in 1963. He received the Ph.D. degree in mathematics from the Institute of Software, the Chinese Academy of Sciences in 1988. He is a professor of the Institute of Software, the Chinese Academy of Sciences. His current research interests include recursion theory, theory of computability, complexity of computation, and rough set theory.

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a is *noncappable*, otherwise.

(ii) Let $\mathbf{M}_{wtt}$ denote the set of all the cappable degrees.

(iii) Let $\mathbf{NC}_{wtt}(=\mathbf{R}_{wtt}-\mathbf{M}_{wtt})$ denote the set of all the noncappable r.e. degrees.

Theorem 1.2.[1] **M** is an ideal in **R**. Moreover, $\{A\in\mathscr{E}:\deg_T(A)\in\mathbf{M}\}=\{A\in\mathscr{E}:\deg_{wtt}(A)\in\mathbf{M}_{wtt}\}$, where $\mathscr{E}$ is the lattice of all the recursively enumerable sets. Therefore, $\mathbf{M}_{wtt}$ is an ideal in $\mathbf{R}_{wtt}$.

Definition 1.3. (i) A coinfinite r.e. set A is *promptly simple* if there are a recursive function q and a recursive enumeration $\{A_s\}_{s\in\omega}$ of A such that for every e,

$$|W_e|=\infty\rightarrow\exists\ x\exists\ s(x\in W_{e,\text{at }s}\ \&\ x\in A_{q(s)}).$$

An r.e. degree **a** is promptly simple if there is a promptly simple r.e. set $A\in\mathbf{a}$.

(ii) Let $\mathbf{PS}_{wtt}$ denote the set of all the promptly simple degrees.

Theorem 1.4. (Promptly Simple Degree Theorem[1]) Let A be an r.e. set. Then A has promptly simple degree iff there are a recursive function q and an enumeration $\{A_s\}_{s\in\omega}$ of A such that for all s, $q(s)\geqslant s$ and for all $e\in\omega$,

$$|W_e|=\infty\rightarrow\exists\ x\exists\ s(x\in W_{e,\text{at }s}\ \&\ A_s\lceil x\neq A_{q(s)}\lceil x), \tag{1.1}$$

namely, A promptly permits some element $x\in W_e$.

Theorem 1.5.[1] NC=PS.

Lemma 1.6. (Slowdown Lemma) Let $\{G_{e,s}\}$ be a strong array of finite sets such that $G_{e,s}\subseteq G_{e,s+1}$ and $G_e=\bigcup_s G_{e,s}$. Then there is a recursive function h such that for all e and s, $W_{h(e)}=G_e$ and $W_{h(e),s}\cap G_{e,\text{at }s}=\varnothing$.

2 Main Theorem and Its Requirements

Theorem 2.1. Given any noncappable r.e. degree $\mathbf{c}<\mathbf{0}'$, there are r.e. degrees **a**, **b** such that $\mathbf{c}<\mathbf{a},\mathbf{b}$; $[\mathbf{c}]<[\mathbf{a}],[\mathbf{b}]$, and $\mathbf{a}\cap\mathbf{b}=\mathbf{c}$.

By the distributivity of $\mathbf{R}_{wtt}$, we have the following.

Corollary 2.2. $[\mathbf{c}]$ is branching in $\mathbf{R}_{wtt}/\mathbf{M}_{wtt}$ for any $[\mathbf{c}]$ such that $[\mathbf{0}]<[\mathbf{c}]<[\mathbf{0}']$.

Proof. Given any $[\mathbf{0}]<[\mathbf{c}]<[\mathbf{0}']$, let $\mathbf{c}\in[\mathbf{c}]$ be an r.e. degree. Then **c** is incomplete and noncappable. By Theorem 2.1, there are r.e. degrees **a** and **b** such that $\mathbf{c}<\mathbf{a},\mathbf{b}$; $[\mathbf{c}]<[\mathbf{a}],[\mathbf{b}]$ and $\mathbf{a}\cap\mathbf{b}=\mathbf{c}$. We now show that $[\mathbf{c}]=[\mathbf{a}]\wedge[\mathbf{b}]$. Let **d** be an r.e. degree such that $[\mathbf{d}]\leq[\mathbf{a}],[\mathbf{b}]$. We prove that $[\mathbf{d}]<[\mathbf{c}]$. Since $[\mathbf{d}]\leq[\mathbf{a}],[\mathbf{b}]$, by the definition of relation $\leq$, there exist r.e. degrees $\mathbf{a}_0,\mathbf{b}_0\in\mathbf{M}_{wtt}$ such that $\mathbf{d}\leqslant\mathbf{a}\cup\mathbf{a}_0$, $\mathbf{b}\cup\mathbf{b}_0$. By the distributivity of $\mathbf{R}_{wtt}$, there exist r.e. degrees $\mathbf{a}^0\leqslant\mathbf{a}$, $\mathbf{a}_0^0\leqslant\mathbf{a}_0$, $\mathbf{b}^0\leqslant\mathbf{b}$ and $\mathbf{b}_0^0\leqslant\mathbf{b}_0$ such that $\mathbf{d}=\mathbf{a}^0\cup\mathbf{a}_0^0=\mathbf{b}_0\cup\mathbf{b}_0^0$. Then $\mathbf{a}_0^0,\mathbf{b}_0^0\in\mathbf{M}_{wtt}$ and $\mathbf{a}^0,\mathbf{b}^0\leqslant\mathbf{a},\mathbf{b}$, so $\mathbf{a}_0^0\cup\mathbf{b}_0^0\in\mathbf{M}_{wtt}$ and $\mathbf{a}^0\cup\mathbf{b}^0\leqslant\mathbf{c}$. Hence, $\mathbf{d}\leqslant\mathbf{c}\cup\mathbf{a}_0^0\cup\mathbf{b}_0^0$, i.e., $[\mathbf{d}]\leq[\mathbf{c}]$.

Corollary 2.3. $\mathbf{R}_{wtt}/\mathbf{M}_{wtt}$ is upper-ward dense.

To prove Theorem 2.1, we fix an r.e. set C such that $[\mathbf{0}]<[\deg_{wtt}(C)]<[\mathbf{0}']$. By Theorems 1.4 and 1.5, assume that C is promptly simple via a recursive function p and fix an enumeration $\{C_s\}$ of C such that for every e,

$$|W_e|=\infty\rightarrow\exists\ x\exists\ s(x\in W_{e,\text{at }s}\ \&\ C_s\lceil x\neq C_{p(s)}\lceil x).$$

We shall recursively construct sets A,B such that $A\not\leqslant_{wtt}C\oplus W_e$ for any cappable r.e. set W_e; $B\not\leqslant_{wtt}C\oplus V_e$ for any cappable r.e. set V_e, and C is the infimum of $A\oplus C$ and $B\oplus C$. Namely the construction will satisfy for every e the following requirements:

$$\mathscr{R}_e:\quad A=\Phi_e(C\oplus U_e)\rightarrow\deg_{wtt}(U_e)\in\mathbf{NC}_{wtt};$$

$$\mathscr{Q}_e:\quad B=\Psi_e(C\oplus V_e)\rightarrow\deg_{wtt}(V_e)\in\mathbf{NC}_{wtt};$$

$$\mathscr{M}_e:\quad \Theta_e(A\oplus C)=\Theta_e(B\oplus C)=f_e\ \text{total}\rightarrow f_e\leqslant_{wtt}C.$$

By Theorems 1.4 and 1.5, we decompose $\mathscr{R}_e$ into the following infinitely many subrequirements: for every

$i\in\omega$,

$$\mathcal{R}_{e,i}: A=\Phi_e(C\oplus U_e)\ \&\ |W_i|=\infty\rightarrow\exists\ x\exists\ s(x\in W_{i,at\ s}\ \&\ U_{e,s}\lceil x\neq U_{e,p_e(s)}\lceil x),$$

where p_e is a recursive function defined in the construction such that if $A=\Phi_e(C\oplus U_e)$, then p_e is total and shows the prompt simplicity of U_e.

Similarly we decompose $\mathcal{Q}_e$ into the following infinitely many subrequirements: for every $i\in\omega$,

$$\mathcal{Q}_{e,i}: B=\Psi_e(C\oplus V_e)\ \&\ |W_i|=\infty\rightarrow\exists\ x\exists\ s(x\in W_{i,at\ s}\ \&\ V_{e,s}\lceil x\neq V_{e,q_e(s)}\lceil x),$$

where q_e is a recursive function defined in the construction such that if $B=\Psi_e(C\oplus V_e)$, then q_e is total and shows the prompt simplicity of V_e.

At any stage $s+1$, we define the length of agreement:

$$l^{\mathcal{R}}(e,s)=\max\{x:\forall\ y<x(A_s(y)=\Phi_{e,s}(C_s\ominus U_{e,s};y))\},$$

$$l^{\mathcal{Q}}(e,s)=\max\{x:\forall\ y<x(B_s(y)=\Psi_{e,s}(C_s\ominus V_{e,s};y))\}.$$

A stage $s+1$ is *$e^{\mathcal{R}}$-expansionary* if $l^{\mathcal{R}}(e,s)>l^{\mathcal{R}}(e,t)$ for every $t<s$.

To satisfy $\mathcal{R}_{e,i}$, at any stage $s+1$, if there exist x and y such that $y\in K_s$, $l^{\mathcal{R}}(e,s)>\langle e,i,y\rangle$, $x\in W_{i,s}-W_{i,s-1}$, and $x>\varphi_e(\langle e,i,y\rangle)$, then enumerate $\langle e,i,y\rangle$, in A, and wait for the next $e^{\mathcal{R}}$-expansionary stage $t+1>s+1$. At $t+1$, define $p_e(s)=t$. If $C_s\lceil\varphi_e(\langle e,i,y\rangle)=C_t\lceil\varphi_e(\langle e,i,y\rangle)$, then $U_{e,s}\lceil\varphi_e(\langle e,i,y\rangle)\neq U_{e,t}\lceil\varphi_e(\langle e,i,y\rangle)$, and $\mathcal{R}_{e,i}$ is satisfied.

To satisfy $\mathcal{M}_e$, at any stage $s+1$, we define the *length of agreement*:

$$l(e,s)=\max\{x:\forall\ y<x(\Theta_{e,s}(A_s\oplus C_s;y)=\Theta_{e,s}(B_s\oplus C_s;y)\downarrow)\}.$$

$s+1$ is e-expansionary if $l(e,s)>l(e,t)$ for every $t<s$. At any e-expansionary stage $t+1$, let $s+1<t$ be the last e-expansionary stage. For any $x<l(e,s)$, if $A_s\lceil\theta_e(x)\neq A_t\lceil\theta_e(x)$ and $B_s\lceil\theta_e(x)\neq B_t\lceil\theta_e(x)$, then we shall ensure that $C_s\lceil\theta_e(x)\neq C\lceil\theta_e(x)$.

3 The Priority Tree and the Basic Module

The priority tree $T=2^{<\omega}$. We define an order $<_L$ on T as follows: for any $\alpha,\beta\in T$,

$$\alpha<_L\beta\leftrightarrow\alpha\subset\beta\vee\exists\ \tau\subseteq\alpha,\beta(\tau\hat{\ }0\subseteq\alpha\ \&\ \tau\hat{\ }1\subseteq\beta).$$

We assign $\mathcal{M}_e$ to α if $|\alpha|=3e$; $\mathcal{R}_e$ to α if $|\alpha|=3e+1$; $\mathcal{Q}_e$ to α if $|\alpha|=3e+2$; $\mathcal{R}_{e,i}$ to α if $|\alpha|=3\langle e,i\rangle+1$; $\mathcal{Q}_{e,i}$ to α if $|\alpha|=3\langle e,i\rangle+2$. We say that α is a *strategy* for the requirements assigned to it.

At any stage $s+1$, we shall define a sequence δ_s of *nodes accessible* at stage $s+1$ as follows.

Let

$$l(\alpha,s)=\begin{cases} l(e,s) & \text{if } |\alpha|=3e,\\ l^{\mathcal{R}}(e,s) & \text{if } |\alpha|=3e+1,\\ l^{\mathcal{Q}}(e,s) & \text{if } |\alpha|=3e+2.\end{cases}$$

We assume that $l(\alpha,0)=0$ for every $\alpha\in T$. For $\alpha\in T$, a stage $s+1$ is an *α-stage* if $\alpha\subseteq\delta_s$ or $s=0$. $s+1$ is α expansionary if $s=0$ or $s+1$ is an α-stage and $l(\alpha,s)>l(\alpha,t)$ for every α-stage $t+1\leqslant s$.

We define an auxiliary function $\tau:T\rightarrow T$ as follows:

$$\tau(\alpha)=\begin{cases}\beta\subseteq\alpha(|\beta|=3e+1) & \text{if } \alpha \text{ is a strategy for } \mathcal{R}_{e,i},\\ \beta\subseteq\alpha(|\beta|=3e+2) & \text{if } \alpha \text{ is a strategy for } \mathcal{Q}_{e,i}.\end{cases}$$

Now we define $\delta_s(n)$ by induction on n for $n<s$. Suppose that $\alpha=\delta_s\lceil n$. $\delta_s(n)=0$ if $s+1$ is an α-expansionary stage, and $\delta_s(n)=1$, otherwise. The *true path* δ is defined by

$$\delta=\liminf_s\delta_s.$$

During the construction some node $\alpha\in T$ will be initialized at certain stage. $\alpha\in T$ is *initialized at stage $s+1$* if every parameter associated with α is set to be undefined.

Let α be a strategy for $\mathscr{R}_{e,i}$. We assign a number z_α to be a follower of α. At any $\tau(\alpha)$-stage $s+1$, if there are an x and a $y \geqslant z_\alpha$ such that

(3.1) $x \in W_{i,s} - W_{i,s'}$, where $s'+1 \leqslant s$ is the last $\tau(\alpha)$-stage,

(3.2) $y \in K_s$, $\langle e,i,y\rangle \notin A_s$, $l(\tau(\alpha),s) > \langle e,i,y\rangle$, and

(3.3) $\varphi_e(\langle e,i,y\rangle) < x$,

then enumerate $\langle e,i,y\rangle$ in A and wait for the next $\tau(\alpha)$-expansionary stage, say $u+1$. At $u+1$, define $p_{\tau(\alpha)}(s) = u$. If $C_s \lceil \varphi_e(\langle e,i,y\rangle) = C_u \lceil \varphi_e(\langle e,i,y\rangle)$, then it must be that $U_{e,s}\lceil \varphi_e(\langle e,i,y\rangle) \neq U_{e,u}\lceil \varphi_e(\langle e,i,y\rangle)$. We say that $\mathscr{R}_{e,i}$ is *satisfied at* $\tau(\alpha)$, and α never enumerates any element in A unless if $\tau(\alpha)$ is initialized.

We use a similar basic module for $\mathscr{Q}_{e',i'}$.

Let β be a strategy for $\mathscr{M}_j$, ξ be a strategy for $\mathscr{Q}_{e',i'}$. Assume that $\beta\hat{\ }0 \subset \tau(\alpha)$. Then some $\langle e,i,y\rangle$ is enumerated in A only at β-expansionary stages. Hence, if $f_j(z)$ is defined at some β-expansionary stage $s+1$ for some z, then $f_j(z)$ does not change because of enumerating $\langle e,i,y\rangle$ in A.

Assume that $\tau(\alpha)\hat{\ }0 \subset \beta\hat{\ }0 \subseteq \alpha,\xi$. Assume that at some $\tau(\alpha)$-stage $s+1$, there is an element $\langle e,i,y\rangle$ to be enumerated in A. Then $s+1$ may not be a β-expansionary stage. If there is some element enumerated in B after the last β-expansionary stage, then $\langle e,i,y\rangle$ enumerated in A may result in $f_j(z)$ changing for some z. Precisely, let $s_\beta+1 \leqslant s$ be the last β-expansionary stage, assume that there are $\langle e',i',y'\rangle \in B_s - B_{s_\beta}$ and a z such that $l(\beta,s_\beta) > z$, $\theta_j(z) > \langle e',i',y'\rangle$. If there are x and y such that (3.1)~(3.3) hold and $\theta_j(z) > \langle e,i,y\rangle$, then we cannot enumerate $\langle e,i,y\rangle$ in A directly. To cope with it, at any $\tau(\alpha)$-stage $s+1$, if there are x and $y \geqslant z_\alpha$ such that (3.1)~(3.3) hold, then define an auxiliary function:

$$z(\alpha,s) = \min\{\theta_j(z) : z < l(\beta,s_\beta), \theta_j(z) > \langle e,i,y\rangle, \exists\ y'(\theta_j(z) > \langle e',i',y'\rangle \in B_s - B_{s_\beta})\},$$

where $s_\beta+1 \leqslant s$ is the last β-expansionary stage. If $z(\alpha,s)$ does not exist, then enumerate $\langle e,i,y\rangle$ in A. Otherwise, firstly enumerate $z(\alpha,s)$ in G_α, secondly enumerate $W_{h(\alpha)}$ until a stage $t \geqslant s$ such that $z(\alpha,s) \in W_{h(\alpha),at\,t}$, and then compute $p(t)$, where h is the recursive function whose existence is guaranteed by the Slowdown lemma. If $C_s\lceil z(\alpha,s) = C_{p(t)}\lceil z(\alpha,s)$, then do nothing; if $C_s\lceil z(\alpha,s) \neq C_{p(t)}\lceil z(\alpha,s)$, then we say that C *promptly permits* $\langle e,i,y\rangle$, enumerate $\langle e,i,y\rangle$ in A and wait for the next α-expansionary stage $u+1 > s$. At $u+1$, define $p_\alpha(s) = u$. If $C_s\lceil \varphi_e(\langle e,i,y\rangle) = C_u\lceil \varphi_e(\langle e,i,y\rangle)$, then $\mathscr{R}_{e,i}$ is satisfied at $\tau(\alpha)$ unless if $\tau(\alpha)$ is initialized afterwards.

Fix any $\langle e,i,y\rangle$ such that $y \geqslant z_\alpha$. If α is on the true path and $|W_i| = \infty$, then there are infinitely many α-stages $s+1$ such that there are x and y satisfying (3.1)~(3.3) at $s+1$. Hence, if $\mathscr{R}_{e,i}$ is not satisfied at $\tau(\alpha)$, then the range of $z(\alpha,s)$, as a function of s, is infinite. By the prompt simplicity of C, there is a stage when $\langle e,i,y\rangle$ is promptly permitted by C and enumerated in A.

If for every y, C changes to below $\varphi_e(\langle e,i,y\rangle)$ after $\langle e,i,y\rangle$ is enumerated in A, then we could show that $K \leqslant_{wtt} C$, a contradiction to the assumption that C is incomplete.

4 Construction

A strategy α for $\mathscr{R}_{e,i}$ *requires attention* at stage $s+1$ if $\mathscr{R}_{e,i}$ is not satisfied at $\tau(\alpha)$, z_α is defined, $\tau(\alpha)\hat{\ }0 \subseteq \alpha$, $\tau(\alpha) \subset \delta_s$ and there exist x and $y \geqslant z_\alpha$ such that

(4.1) $x \in W_{i,s} - W_{i,s'}$, where $s'+1 \leqslant s$ is the last $\tau(\alpha)$-stage,

(4.2) $y \in K_s$, $\langle e,i,y\rangle < l(\tau(\alpha),s)$, $\langle e,i,y\rangle \notin A_s$,

(4.3) $\varphi_e(\langle e,i,y\rangle) < x$, and

(4.4) there is an α-stage $\leqslant s$ after the last α-stage when α received attention.

It is similar to defining a strategy α for $\mathscr{Q}_{e,i}$ *requiring attention* at stage $s+1$.

Stage $s=0$: Set $A_0 = B_0 = \varnothing$ and initialize every $\alpha \in T$.

Stage $s+1$: The construction will proceed by performing the following steps.

Step 1. For every strategy α with $\alpha\hat{}\ 0\subset\delta_s$, if α is a strategy for $\mathscr{R}_e$(or $\mathscr{Q}_e$), then define $p_\alpha(s')=s$ for every $s'\leqslant s$ with $p_{\alpha,s}(s')\uparrow$, and if there is a strategy β for $\mathscr{R}_{e,i}$(or $\mathscr{Q}_{e,i}$) such that $\tau(\beta)=\alpha$ and some $\langle e,i,y\rangle$ was enumerated in A (or B) after the last α-expansionary stage $t+1$ and $C_t\lceil\varphi_e(\langle e,i,y\rangle)=C_s\lceil\varphi_e(\langle e,i,y\rangle)$ (or $C_t\lceil\psi_e(\langle e,i,y\rangle)=C_s\lceil\psi_e(\langle e,i,y\rangle)$), then $\mathscr{R}_{e,i}$(or $\mathscr{Q}_{e,i}$) is satisfied at α unless α is initialized afterwards.

Step 2. Find the least α requiring attention at stage $s+1$.

Case 1. If α is a strategy for $\mathscr{R}_{e,i}$, then let y be the least one satisfying (4.1)～(4.3). If for every $\mathscr{M}$-strategy β with $\tau(\alpha)\subset\beta\hat{}\ 0\subseteq\alpha$, there is no element enumerated in B by any $\xi\supset\beta$ after the last β-expansionary stage, then go to step 2. Otherwise, firstly define

$$z(\alpha,s)=\min\{\theta_\beta(z):\tau(\alpha)\subset\beta\hat{}\ 0\subseteq\alpha, z<l(\beta,s_\beta), \langle e,i,y\rangle<\theta_\beta(z), \exists\ y'(\theta_\beta(z)>y'\in B_s-B_{s_\beta})\},$$

where $s_\beta+1\leqslant s$ is the last β-expansionary stage and $\theta_\beta(y)$ is the recursive use function of $\Theta_j(A\oplus C)$ if β is a strategy for $\mathscr{M}_j$; secondly enumerate $z(\alpha,s)$ in G_α, enumerate $W_{h(\alpha)}$ until a stage $t\geqslant s$ such that $z(\alpha,s)\in W_{h(\alpha),\text{at } t}$, and then compute $p(t)$. If $C_s\lceil z(\alpha,s)\neq C_{p(t)}\lceil z(\alpha,s)$, then go to step 3, otherwise, go to step 4.

Case 2. If α is a strategy for $\mathscr{Q}_{e,i}$, then proceed as in case 1 replacing A by B and B by A.

Step 3. Enumerate $\langle e,i,y\rangle$ in A if α is a strategy for $\mathscr{R}_{e,i}$; enumerate in B if α is a strategy for $\mathscr{Q}_{e,i}$; and initialize every node $\gamma>_L\alpha$.

Step 4. For every $\alpha\subset\delta_s$, if z_α is undefined, then assign the least unused number to be z_α.

Step 5. Initialize every node $\gamma>_L\delta_s$.

This ends the description of the construction.

5 Verification

Let $\delta=\liminf_s \delta_s$ be the *true path*. We prove the following lemmas by induction on $\alpha\subset\delta$.

Lemma 5.1. If $\alpha\subset\delta$ is a strategy for $\mathscr{R}_{e,i}$, such that $\tau(\alpha)\hat{}\ 0\subset\delta$, then $\mathscr{R}_{e,i}$ is satisfied at $\tau(\alpha)$ eventually and α enumerates only finitely many elements in A.

Proof. Let s_0 be the least α-stage such that α is not initialized after s_0. Let z_α be the follower of α at the end of stage s_0. Since $\tau(\alpha)\hat{}\ 0\subset\delta$, we have $A=\Phi_e(C\oplus U_e)$.

Assume that $\mathscr{R}_{e,i}$ is not satisfied at $\tau(\alpha)$, then $|W_i|=\infty$. There are infinitely many $\tau(\alpha)$-stages $s+1$ when α requires attention. Given any $y\geqslant z_\alpha$ with $y\in K$, if $\langle e,i,y\rangle\notin A$, then there are infinitely many $\tau(\alpha)$-stages $s+1$ when α requires attention via y, and so the range of $z(\alpha,s)$, as a function of s, is infinite. By the prompt simplicity of C, there exist an x and a stage $s+1\geqslant s_0$ such that α requires attention via x, $\langle e,i,y\rangle$ is enumerated in A at $s+1$, a contradiction. Hence, $\langle e,i,y\rangle$ is enumerated in A at some $\tau(\alpha)$-stage, say $s+1$, and $C_s\lceil\varphi_e(\langle e,i,y\rangle)\neq C_t\lceil\varphi_e(\langle e,i,y\rangle)$, otherwise $\mathscr{R}_{e,i}$ would be satisfied at $\tau(\alpha)$, where $t+1\geqslant s$ is the next $\tau(\alpha)$-expansionary stage. Now we show that $K\leqslant_{wtt}C$. To decide whether $y\in K$ for any given $y\geqslant z_\alpha$, find a stage $s+1>s_0$ such that $C_s\lceil\varphi_e(\langle e,i,y\rangle)=C\lceil\varphi_e(\langle e,i,y\rangle)$, then $y\in K$ iff $y\in K_s$. This is a contradiction to the assumption that C is incomplete. Hence, $\mathscr{R}_{e,i}$ is satisfied at $\tau(\alpha)$ eventually, and α enumerates finitely many elements in A.

Similarly we can prove the following.

Lemma 5.2. If $\alpha\subset\delta$ is a strategy for $\mathscr{Q}_{e,i}$ such that $\tau(\alpha)\hat{}\ 0\subset\delta$, then $\mathscr{Q}_{e,i}$ is satisfied at $\tau(\alpha)$ eventually and α enumerates only finitely many elements in B.

Lemma 5.3. If $\alpha\subset\delta$ is a strategy for $\mathscr{M}_e$, then $\mathscr{M}_e$ is satisfied.

Proof. Let s_0 be the stage as defined in Lemma 5.1. If $\alpha\hat{}\ 0\not\subset\delta$, then $\mathscr{M}_e$ is satisfied; otherwise, f_e is total. To C-recursively compute $f_e(y)$ for any given y, find an α-expansionary stage $s+1\geqslant s_0$ such that

$$C_s\lceil\theta_e(y)=C\lceil\theta_e(y),$$

then $f_e(y)=f_{e,s}(y)$. We now show that at any α-expansionary stage $t+1>s$, if $A_t\lceil\theta_e(y)\neq A_{t'}\lceil\theta_e(y)$ and $B_t\lceil\theta_e(y)\neq B_{t'}\lceil\theta_e(y)$, where $t'+1>t$ is the next α-expansionary stage, then $C_t\lceil\theta_e(y)\neq C\lceil\theta_e(y)$. By the construction, we assume that some $z<\theta_e(y)$ is enumerated in A at $t+1$. Then by the initialization, another $z'<\theta_e(y)$ is enumerated in B by any $\mathcal{Q}$-strategy ξ at any stage $u+1\leqslant t'$, which $\geqslant t$ only if $\tau(\xi)\subset\alpha\hat{}\ 0\subset\xi$ and C promptly permits $\theta_e(y)$ at $u+1$, i.e., $C_t\lceil\theta_e(y)\neq C\lceil\theta_e(y)$, a contradiction to the choice of s. Hence, $f_{e,t}(y)=f_{e,t'}(y)$.

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$\mathbf{R}_{wtt}/\mathbf{M}_{wtt}$中的枝点元素

张再跃 眭跃飞

(扬州大学工业学院计算机系 扬州 225009)

(中国科学院软件研究所 北京 100080)

摘要 该文证明了在$\mathbf{R}_{wtt}/\mathbf{M}_{wtt}$中除了最大元和最小元外,每个元c是枝点元素,即为某两个大于c的元素的最大下界,其中$\mathbf{R}_{wtt}/\mathbf{M}_{wtt}$是递归可枚举弱真值表归约度集$\mathbf{R}_{wtt}$模可盖递归可枚举弱真值表归约度集$\mathbf{M}_{wtt}$的商.

关键词 递归可枚举度,弱真值表归约.

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