ISSN 1000-9825, CODEN RUXUEW
Journal of Software, Vol.19, No.12, December 2008, pp.3170−3178
DOI: 10.3724/SP.J.1001.2008.03170

E-mail: jos@iscas.ac.cn
http://www.jos.org.cn
Tel/Fax: +86-10-62562563

基于一阶模态逻辑的模糊推理*

张晓如[1+]，张再跃[1]，眭跃飞[2]，黄智生[3]

[1](江苏科技大学 计算机科学与工程学院 智能信息处理联合实验室,江苏 镇江 212003)

[2](中国科学院 计算技术研究所 智能信息处理重点实验室,北京 100190)

[3](Department of Artificial Intelligence, Vrije University of Amsterdam, Netherlands)

Fuzzy Reasoning Based on First-Order Modal Logic

ZHANG Xiao-Ru[1+], ZHANG Zai-Yue[1], SUI Yue-Fei[2], HUANG Zhi-Sheng[3]

[1](Joint Laboratory of Intelligent Information Processing, School of Computer Science and Engineering, Jiangsu University of Science and Technology, Zhenjiang 212003, China)

[2](Key Laboratory of Intelligent Information Processing, Institute of Computing Technology, The Chinese Academy of Sciences, Beijing 100190, China)

[3](Department of Artificial Intelligence, Vrije University of Amsterdam, Netherlands)

+ Corresponding author: E-mail: njzzy@yzcn.net

Zhang XR, Zhang ZY, Sui YF, Huang ZS. Fuzzy reasoning based on first-order modal logic. *Journal of Software*, 2008,19(12):3170−3178. http://www.jos.org.cn/1000-9825/19/3170.htm

Abstract: As an extension of traditional modal logics, this paper proposes a fuzzy first-order modal logic based on beilievable degree, and gives out a description of the fuzzy first-order modal logic based on constant domain semantics. In order to make the reasoning procedure between the fuzzy assertions efficiently, the notion of the fuzzy constraint is considered. A fuzzy constraint is an expression in which both syntax ingredient and semantics information are contained. By using the notion of the constraint, the reasoning procedure between the fuzzy assertions can be directly considered in the semantics environment, thus a fuzzy reasoning formal system which contains fuzzy constraint as its basic element is developed. As a main work of the paper, the relationship between the validity of the new assertion and the satisfiability of the fuzzy constraints is analyzed, and reasoning rules of the fuzzy reasoning formal system based on first order modal logic are given out. Further work could be done by considering the soundness and completeness of the formal system, and by building an efficient mechanism of reasoning procedure. The results have potential application in the areas of artificial intelligence and computer science.

Key words: modal logic; fuzzy reasoning; formal system; fuzzy constraint; satisfiability

---

* Supported by the National Natural Science Foundation of China under Grant Nos.60573063, 60573064 (国家自然科学基金); the National Basic Research Program of China under Grant Nos.G1999032701, 2003CB317008 (国家重点基础研究发展计划(973)); the National Laboratory of Intelligent Information Processing of China (国家智能信息处理重点实验室); the National Laboratory of Software Development Environment of China (国家软件发展环境重点实验室)

Received 2007-11-18; Accepted 2008-07-03

摘 要: 研究基于可信度的模糊一阶模态逻辑,给出了基于常域的模糊一阶模态逻辑语义以及推理形式系统描述.为有效进行模糊断言间的推理,考虑了模糊约束的概念.模糊约束是一个表达式,其中既有语法成分又包含意义信息.模糊推理形式系统中的基本对象是模糊约束,针对模糊约束引进可满足性概念,研究模糊约束可满足性相关性质.利用模糊约束的概念,模糊断言间的推理可以直接在语义环境下加以考虑,因此,以模糊约束为基本元素的模糊推理形式系统随之建立.主要分析新产生断言有效性与模糊约束集可满足性之间的关系,并在此基础上给出了模糊推理形式系统的推理规则.进一步的工作可探讨模糊推理形式系统的可靠性与完全性,建立推理过程的能行机制.研究结果可在人工智能和计算机科学等领域得以应用.

关键词: 模态逻辑;模糊推理;形式系统;模糊约束;可满足性
中图法分类号: TP18 **文献标识码**: A

1 Introduction

Modal logic[1] is considered as an important logic branch, which was developed firstly in the category of nonclassical logics, and has been now widely used as a formalism for knowledge representation in artificial intelligence and an analysis tool in computer science[2,3]. Along with the study of modal logics, it has been found that modal logics have a close relationship with many other knowledge representation formalisms. The most well-known result is the connection of the possible world semantics for the modal epistemic logic S_5 with the approximation space in the rough set theory[4], where the system S_5 has been shown to be useful in the analysis of knowledge in various areas[5]. As a fragment of first order logic, modal logics are limited to deal with crisp assertions, as its possible world semantics are crisp. That is, assertions about whether a formal proposition holds is a yes-no question. More often than not, the assertions encountered in the real world are not precise and thus cannot be treated simply by using ‘yes’ or ‘no’. The fuzzy logic directly deals with the notion of vagueness and imprecision, and has been used in many research areas such as interval mathematics[6], possibility theory[7], rough set theory or artificial neural networks. By combining with the fuzzy logic, traditional modal logic has been extended. For example, Hájek[8] provided a complete axiomatization of fuzzy S_5 system, where the accessibility relation is a universal relation. Godo and Rodríguez[9,10] gave a complete axiomatic system for the extension of Hájec’s logic with another modality corresponding to a fuzzy similarity relation. Zhang, *et al*.[11,12] established a formal system of fuzzy reasoning based on propositional modal logics and discussed the soundness and completeness of the system. Subsequently they also gave out a description of the fuzzy first-order modal logic based on constant domain semantics[13]. The work in this paper is an extensive study of the fuzzy reasoning based on first-order modal logic. We shall discuss the properties of the fuzzy first-order modal logic based on constant domain semantics, study the relationship of the reasoning procedures between the classical modal logic and the fuzzy modal logic, introduce a fuzzy reasoning formal system based on fuzzy first-order modal logics and give out a description of the fuzzy reasoning process based on the satisfiability.

2 A Quick Overview of First-Order Modal Logic

In general, the alphabet of a first-order modal logic consists of the following symbols: a set of variable symbols, denoted by $VS=\{x_1,x_2,\ldots\}$; a set of relation symbols, denoted by $PS=\{P_i^n : n=1,2,\ldots\}$, where P_i^n is the ith n-place relation symbol; the logical symbols, $\neg$ (negation), $\wedge$ (and), $\vee$ (or), $\supset$ (material implication); quantifiers $\forall$ (for all) and $\exists$ (exists); the modal operator symbols $\Box$ (necessity operator) and $\Diamond$ (possibility operator).

Definition 2.1. An *atomic formula* of the first-order modal logic is an expression of the form $P(x_1,\ldots,x_n)$, where P is an n-place relation symbol and $x_1,\ldots,x_n$ are variables.

Definition 2.2. The set of *first-order formulas* of a first-order modal logic is the smallest set satisfying the following conditions: Every atomic formula is a formula; if φ is a formula, so are $\neg\varphi$, $\Box\varphi$, $\Diamond\varphi$, $\forall x\varphi$ and $\exists x\varphi$; if φ and ψ are formulas and $\circ$ is a binary connective, then $\varphi\circ\psi$ is a formula.

When establishing the formal systems of modal logics, it was convenient to take $\neg$, $\supset$ and $\Box$ as primitive, and the other connectives and modal operator are defined as usual. For quantifiers we take $\forall$ as primitive, and treat $\exists$ as a defined one. Hence, our modal logic formal system contains following axioms and inference rules:

• Axioms:

*Ap*1 $(\varphi\supset(\psi\supset\varphi))$;

*Ap*2 $(((\varphi\supset(\psi\supset\gamma))\supset((\varphi\supset\psi)\supset(\varphi\supset\gamma)))$;

*Ap*3 $((\neg\varphi\supset\neg\psi)\supset(\psi\supset\varphi))$;

*Ap*4 $(\forall x\ \varphi(x)\supset\varphi(y))$, where y is any variable free for x in $\varphi(x)$;

*Ap*5 $(\forall x(\varphi\supset\psi)\supset(\forall x\ \varphi\supset\forall x\ \psi))$;

K $(\Box(\varphi\supset\psi)\supset(\Box\varphi\supset\Box\psi))$;

T $(\Box\varphi\supset\varphi)$;

E $(\neg\Box\neg\varphi\supset\Box\neg\Box\neg\varphi)$.

• Inference rules:

N (necessity rule) if $\vdash\varphi$ then $\vdash\Box\varphi$;

UG (universal generalization) if $\vdash\varphi$ then $\vdash\forall x\varphi$;

MP (modus ponens) if $\vdash\varphi\supset\psi$ and $\vdash\varphi$ then $\vdash\psi$.

Definition 2.3. A *reasoning process* in the formal system of the first-order modal logic is a sequence of formulas $\varphi_1,\ldots,\varphi_n$ such that for each $i(1\le i\le n)$, either φ_i is an axiom of the formal system or φ_i follows from previous members of the sequence by the inference rules *N* or *UG* or *MP*. The last member φ_n of the sequence is said to be a *theorem* of the formal system.

A *constant domain semantics* (or a *model*) of a first-order modal logic is a structure $M=\langle W,R,D,I\rangle$, where W is a set of possible worlds, R is a relation on W, D is a non-empty set called *domain* of the frame $\langle W,R\rangle$, I is an *interpretation* of the frame $\langle W,R,D\rangle$, which assigns to each n-place relation symbol P and to each possible world $w\in W$, some n-place relation on the domain D of the frame. Thus, $I(P,w)$ is an n-place relation on D, and so each n-tuple $\langle d_1,\ldots,d_n\rangle$ of members of D either is in the relation $I(P,w)$ or is not. Notice that in a constant domain semantics, the domain of quantification is the same from world to world.

Remark: Different modal logics are characterized by different classes of frames which rely on the properties of the relations defined between the possible worlds. Without loss of generality, we shall consider the logic which is characterized by the class of reflexive, symmetric and transitive frames.

Let $M=\langle W,R,D,I\rangle$ be a constant domain model of a first-order modal logic. A *valuation* in the model M is a mapping v that assigns to each free variable x some member $v(x)$ of the domain D of the model.

Definition 2.4. Let M be a constant domain model of a first-order modal logic and φ be a formula. For each $w\in W$ and each valuation v in M, the notation that the formula φ is true at world w of model M with respect to valuation v, denoted by $M,w|=_v\varphi$, will be defined as follows:

(1) If φ is an atomic formula $P(x_1,\ldots,x_t)$, then $M,w|=_v P(x_1,\ldots,x_t)$ provided $\langle v(x_1),\ldots,v(x_t)\rangle\in I(P,w)$.

(2) $M,w|=_v\neg\varphi\Leftrightarrow M,w|\neq_v\varphi$.

(3) $M,w|=_v\varphi\supset\psi\Leftrightarrow M,w|=_v\neg\varphi$ or $M,w|=_v\psi$.

(4) $M,w|=_v\Box\varphi\Leftrightarrow$ for every $w'\in W$, if wRw' then $M,w'|=_v\varphi$.

(5) $M,w|=_v\forall x\varphi\Leftrightarrow$ for every x-variant v' of v in M, $M,w|=_{v'}\varphi$, where v' is an x-variant of v means that v' and v agree on all variables except possibly the variable x.

Definition 2.5. Let M be a constant domain model of a first-order modal logic and φ be a formula. For each $w\in W$, we say that φ is true at world w of model M, denoted by $M,w|=\varphi$, if $M,w|=_v\varphi$ for every valuation v in M; we say that φ is true in model M, denoted by $M|=\varphi$, if $M,w|=\varphi$ for every world w of M.

Proposition 2.6. For any formula φ of the first-order modal logic, if φ is a theorem of the formal system then φ is true in every model $M=\langle W,R,D,I\rangle$ of the system such that R is an equivalent relation on W.

Usually, first-order modal logics are limited to deal with crisp concepts. However, many useful concepts encountered in the real world do not have a precisely defined criteria for the membership. To cope with this, we shall introduce a fuzzy first-order modal logic system based on *believable degrees*.

3 Fuzzy First-Order Modal Logic with Believable Degrees

Our fuzzy first-order modal system will use the same alphabet of symbols and the set of formulas as that in the first-order modal logic system mentioned in Section 2. In order to deal with vague notions, we extend the first-order modal logics by using expressions of the form $\langle\varphi(x_1,...,x_t),\varepsilon\rangle$ with the intended meaning that the believable degree of the individuals expressed by the variables $x_1,...,x_t$ having the relation expressed by the formula φ is at least ε, where $\varphi(x_1,...,x_t)$ is a formula of the first-order modal logic, and $\varepsilon\in[0,1]$. An expression of the form $\langle\varphi,\varepsilon\rangle$ is also called a fuzzy assertion.

Definition 3.1. A constant domain semantics (or a model) of a fuzzy first-order modal logic is a structure $\mathcal{M}=\langle\mathcal{W},\mathcal{R},\mathcal{D},\mathcal{I}\rangle$ where $\mathcal{W}$ is a set of possible worlds, $\mathcal{R}$ is an equivalence relation on $\mathcal{W}$, $\mathcal{D}$ is a non-empty set called the domain of the frame $\langle\mathcal{W},\mathcal{R}\rangle$, $\mathcal{I}$ is an interpretation of the frame $\langle\mathcal{W},\mathcal{R},\mathcal{D}\rangle$, which assigns to each formula $\varphi(x_1,...,x_t)$ with free variables $x_1,...,x_t$ and to each possible world $w\in\mathcal{W}$, some t-place function on the domain $\mathcal{D}$ of the frame such that following conditions are satisfied:

(1) If φ is an atomic formula $P(x_1,...,x_t)$ then for each t-tuple $\langle d_1,...,d_t\rangle$ of members of $\mathcal{D}$, $\mathcal{I}(P,w)(d_1,...,d_t)\in[0,1]$.

(2) $\mathcal{I}(\neg\varphi,w)(d_1,...,d_t)=1-\mathcal{I}(\varphi,w)(d_1,...,d_t)$.

(3) $\mathcal{I}(\varphi\supset\psi,w)(d_1,...,d_t)=\max\{1-\mathcal{I}(\varphi,w)(d_1,...,d_t),\mathcal{I}(\psi,w)(d_1,...,d_t)\}$.

(4) $\mathcal{I}(\Box\varphi,w)(d_1,...,d_t)=\inf\{\mathcal{I}(\varphi,w')(d_1,...,d_t): w\,\mathcal{R}\,w'\}$.

(5) For any $\varphi(x,x_1,...,x_t)$ and any t-tuple $\langle d_1,...,d_t\rangle$ of members of $\mathcal{D}$, $\mathcal{I}(\forall x\varphi,w)(d_1,...,d_t)=\inf\{\mathcal{I}(\varphi,w)(d,d_1,...,d_t): d\in\mathcal{D}\}$.

Let $\mathcal{M}=\langle\mathcal{W},\mathcal{R},\mathcal{D},\mathcal{I}\rangle$ be a fuzzy constant domain model. An evaluation in the model $\mathcal{M}$ is a mapping v that assigns to each free variable x some member $v(x)$ of the domain $\mathcal{D}$ of the model.

Definition 3.2. Let $\mathcal{M}$ be a constant domain model of a first-order modal logic and φ be a formula with free variables $x_1,...,x_t$. For each $w\in\mathcal{W}$ and each valuation v in $\mathcal{M}$, a fuzzy assertion $\langle\varphi,\varepsilon\rangle$ is true at world w of model $\mathcal{M}$ with respect to valuation v, denoted by $\mathcal{M},w|\approx_v\langle\varphi,\varepsilon\rangle$, if $\mathcal{I}(\varphi,w)(v(x_1),...,v(x_t))\geq\varepsilon$.

Proposition 3.3. Let $\mathcal{M}$ be a constant domain model of a first-order modal logic and φ be a formula. Then for each $w\in\mathcal{W}$ and each valuation v in $\mathcal{M}$, the following properties hold.

(1) If φ is an atomic formula $P(x_1,...,x_t)$, then $\mathcal{M},w|\approx_v\langle P(x_1,...,x_t),\varepsilon\rangle$ provided $\mathcal{I}(P,w)(v(x_1),...,v(x_t))\geq\varepsilon$.

(2) If $\mathcal{M},w|\approx_v\langle\neg\varphi,\varepsilon\rangle$ then $\mathcal{I}(\varphi,w)(v(x_1),...,v(x_t))\leq 1-\varepsilon$.

(3) $\mathcal{M},w|\approx_v\langle\varphi\supset\psi,\varepsilon\rangle\Leftrightarrow\mathcal{M},w|\approx_v\langle\neg\varphi,\varepsilon\rangle$ or $\mathcal{M},w|\approx_v\langle\psi,\varepsilon\rangle$.

(4) $\mathcal{M},w|\approx_v\langle\Box\varphi,\varepsilon\rangle\Leftrightarrow$ for every $w'\in\mathcal{W}$, if $w\mathcal{R}w'$ then $\mathcal{M},w'|\approx_v\langle\varphi,\varepsilon\rangle$.

(5) $\mathcal{M},w|\approx_v\langle\forall x\,\varphi,\varepsilon\rangle\Leftrightarrow$ for every x-variant v' of v in $\mathcal{M}$, $\mathcal{M},w|\approx_{v'}\langle\varphi,\varepsilon\rangle$.

Definition 3.4. Let $\mathcal{M}$ be a constant domain model of a fuzzy first-order modal logic and $\langle\varphi,\varepsilon\rangle$ be a fuzzy assertion. For each $w\in\mathcal{W}$, we say that $\langle\varphi,\varepsilon\rangle$ is true at world w of model $\mathcal{M}$, denoted by $\mathcal{M},w|\approx\langle\varphi,\varepsilon\rangle$, if

$\mathcal{M},w|\approx_v\langle\varphi,\varepsilon\rangle$ for every valuation v in $\mathcal{M}$; we say that $\langle\varphi,\varepsilon\rangle$ is true in model $\mathcal{M}$, denoted by $\mathcal{M}|\approx\langle\varphi,\varepsilon\rangle$, if $\mathcal{M},w|\approx\langle\varphi,\varepsilon\rangle$ for every world w of $\mathcal{M}$.

Proposition 3.5. Let $\mathcal{M}$ be a model and $\varphi(x)$ be a formula in which x is free, and let y be any variable which is free for x in $\varphi(x)$. Suppose v and v' are valuations in $\mathcal{M}$ such that v' is x-variant of v and has $v'(x)=v(y)$. For any $\varepsilon\in[0,1]$ and any possible world w, $\mathcal{M},w|\approx_v\langle\varphi(y),\varepsilon\rangle$ if and only if $\mathcal{M},w|\approx_{v'}\langle\varphi(x),\varepsilon\rangle$.

Remark 1: Proposition 3.5 describes a basic principle of the substitution for variables by other variables. It is an important technique that will be used later in the section where the satisfiability of the fuzzy constraints is discussed.

Proposition 3.6.[13] Let $\mathcal{M}=\langle\mathcal{W},\mathcal{R},\mathcal{D},\mathcal{I}\rangle$ be any fuzzy first-order modal model such that $\mathcal{R}$ is an equivalence relation on $\mathcal{W}$, we have the following properties:

(a) For any axiom φ of the first-order modal logic, $\mathcal{M}|\approx\langle\varphi,0.5\rangle$;

(b) If $\mathcal{M}|\approx\langle\varphi,\varepsilon\rangle$ then $\mathcal{M}|\approx\langle\Box\varphi,\varepsilon\rangle$.

(c) If $\mathcal{M}|\approx\langle\varphi\supset\psi,\varepsilon\rangle$ and $\mathcal{M}|\approx\langle\varphi,\varepsilon'\rangle$ then $\mathcal{M}|\approx\langle\psi,\varepsilon\rangle$, where $\varepsilon,\varepsilon'\in[0,1]$ such that $\varepsilon>1-\varepsilon'$.

(d) If $\mathcal{M}|\approx\langle\varphi,\varepsilon\rangle$ then $\mathcal{M}|\approx\langle\forall x\varphi(x),\varepsilon\rangle$.

In the formal system of the first-order modal logic, a reasoning process will guarantee that every theorem is true in every model $M=\langle W,R,D,I\rangle$ such that R is an equivalent relation on W. In the fuzzy system, we have known by Proposition 3.6 that the believable degree of any axiom is at lest 0.5. It is natural to ask whether or not the believable degree of the theorem in the formal system is also ≥ 0.5 in the fuzzy system? The following proposition partially answers the question.

Proposition 3.7. Let $\mathcal{M}=\langle\mathcal{W},\mathcal{R},\mathcal{D},\mathcal{I}\rangle$ be any fuzzy first-order modal model such that $\mathcal{R}$ is an equivalence relation on $\mathcal{W}$. If $\mathcal{M}|\approx\langle\varphi,0.5^+\rangle$ for any axiom φ in the formal system of the first-order modal logic, then $\mathcal{M}|\approx\langle\psi,0.5^+\rangle$ for any theorem ψ in the formal system of the first-order modal logic, where 0.5^+ means that the believable degree of the relative formula is >0.5.

4 The Relationship Between the General Reasoning and the Fuzzy One

Let Σ be a set of formulas of the first-order modal logic and φ be a formula. We say that φ is a *logical consequence* of Σ, denoted by $\Sigma\models\varphi$, if every model of Σ is a model of φ. In a fuzzy first-order modal logic, a set of basic fuzzy assertions is called a *fuzzy knowledge base*. Assume that $\tilde{\Sigma}$ is a fuzzy knowledge base and $\langle\varphi,\varepsilon\rangle$ is a fuzzy assertion. We say that $\langle\varphi,\varepsilon\rangle$ is a *logical consequence* of $\tilde{\Sigma}$, denoted by $\tilde{\Sigma}|\approx\langle\varphi,\varepsilon\rangle$, if every model of $\tilde{\Sigma}$ is a model of $\langle\varphi,\varepsilon\rangle$.

Definition 4.1. If $\tilde{\Sigma}$ is a fuzzy knowledge base then the crisp knowledge base with respect to $\tilde{\Sigma}$ is a set of formulas Σ such that for any φ, $\varphi\in\Sigma$ if and only if $\langle\varphi,\varepsilon\rangle\in\tilde{\Sigma}$ for some $\varepsilon\in[0,1]$.

To study the relations between $\models$ and $|\approx$, we consider the relations between the general semantics and the fuzzy semantics of the first-order modal logic. Generally, the crisp semantics can be viewed as a special case of the fuzzy semantics. If $\tilde{\Sigma}$ is a fuzzy knowledge base and Σ is the crisp one with respect to $\tilde{\Sigma}$, then it is obvious that every crisp model of Σ is also a model of $\tilde{\Sigma}$, thus we immediately have the following theorem.

Theorem 4.1. Let $\tilde{\Sigma}$ be a fuzzy knowledge base and Σ be the crisp one with respect to $\tilde{\Sigma}$. For any φ and any $\varepsilon>0$, if $\tilde{\Sigma}|\approx\langle\varphi,\varepsilon\rangle$ then $\Sigma\models\varphi$.

The converse of Theorem 4.1 is not true in the general case. A closer relationship holds whenever we consider normalized fuzzy knowledge bases.

Definition 4.2. A fuzzy assertion $\langle\varphi,\varepsilon\rangle$ is said to be *normalized* if $\varepsilon>0.5$. A fuzzy knowledge base is normalized if every fuzzy assertion in it is.

Let $\mathcal{M}=\langle \mathcal{W},\mathcal{R},\mathcal{D},\mathcal{I}\rangle$ be a fuzzy model. The crisp semantics with respect to $\mathcal{M}$, is a quadruple of the form $M=\langle W,R,D,I\rangle$, such that $W=\mathcal{W}$, $R=\mathcal{R}$, $D=\mathcal{D}$ and for any atomic formula $P(x_1,...,x_t)$ and any possible world $w\in W$, $I(P,w)(d_1,...,d_t)=1$ if and only if $\mathcal{I}(P,w)(d_1,...,d_t)>0.5$, where $(d_1,...,d_t)$ is any t-tuple of D.

By induction on the connectives and modal words in φ, we can prove the following lemma.

Lemma 4.2. Let $\mathcal{M}=\langle \mathcal{W},\mathcal{R},\mathcal{D},\mathcal{I}\rangle$ be a fuzzy model of the first-order modal logic and $M=\langle W,R,D,I\rangle$ be the crisp one with respect to $\mathcal{M}$. For any possible world $w\in\mathcal{W}$ and any formula $\varphi(x_1,...,x_t)$ of the first-order modal logic, if $\mathcal{I}(\varphi,w)(v(x_1),...,v(x_t))>0.5$, then we have that $I(\varphi,w)(v(x_1),...,v(x_t))=1$ in M; if $\mathcal{I}(\varphi,w)(v(x_1),...,v(x_t))<0.5$, then $I(\varphi,w)(v(x_1),...,v(x_t))=0$, where v is any valuation in $\mathcal{M}$.

Theorem 4.3. Let $\widetilde{\Sigma}$ be a normalized knowledge base and Σ be the crisp one with respect to $\widetilde{\Sigma}$. If $\Sigma\models\varphi$ then there is an $\varepsilon\geq 0.5$ such that $\widetilde{\Sigma}\mid\approx\langle\varphi,\varepsilon\rangle$.

Proof: If it is not the case, then we would have a model $\mathcal{M}$ which is a model of $\widetilde{\Sigma}$ and there is a possible world $w_0\in\mathcal{W}$ and a valuation v_0 in $\mathcal{M}$ such that $\mathcal{I}(\varphi,w_0)(v_0(x_1),...,v_0(x_t))<0.5$. Since $\mathcal{M}$ is a model of $\widetilde{\Sigma}$, we have that for any $\psi\in\Sigma$, any $w\in\mathcal{W}$ and any valuation v in $\mathcal{M}$, $\mathcal{I}(\psi,w)(v(x_1),...,v(x_t))>0.5$. Let M be the crisp model with respect to $\mathcal{M}$. By Lemma 4.2, M is a model of Σ but not a model of φ, which contradicts that $\Sigma\models\varphi$. □

5 Fuzzy Reasoning and Satisfiability

Let $\widetilde{\Sigma}$ be a fuzzy knowledge base and $\langle\varphi,\varepsilon\rangle$ be a fuzzy assertion. The process of deciding whether $\widetilde{\Sigma}\mid\approx\langle\varphi,\varepsilon\rangle$ or not is called a *fuzzy reasoning process based on fuzzy first-order modal logics*. To verify that $\widetilde{\Sigma}\mid\approx\langle\varphi,\varepsilon\rangle$, one has to verify that every model of $\widetilde{\Sigma}$ is a model of $\langle\varphi,\varepsilon\rangle$, which is not convenient in practical applications. To cope with it, we attempt to find a method which can be used to decide whether or not $\widetilde{\Sigma}\mid\approx\langle\varphi,\varepsilon\rangle$ effectively. The associated work about fuzzy propositional modal logics has been discussed in Refs.[11,12]. In the following, we shall extend the work from propositional modal logic to first-order logic and develop a formal system of fuzzy reasoning based on fuzzy first-order modal logic.

The basic idea is to syntactize the semantic information. More precisely we will extend the logical language for the fuzzy first-order modal logic in a way such that there are infinitely many possible world symbols $\boldsymbol{w}$ and a binary relation symbol $\mathfrak{R}$. Under an extended interpretation $\mathcal{I}$, $\boldsymbol{w}$ is interpreted as a possible world, and $\mathfrak{R}$ as R, the accessibility relation in the frame of the interpretation $\mathcal{I}$.

Definition 5.1. In addition to the basic symbols of the first-order modal logic, the formal system of fuzzy reasoning also contains a set of possible worlds symbols $\boldsymbol{w}_1,\boldsymbol{w}_2,...$, a set of relation symbols $\{<,\leq,>,\geq\}$ and a binary relation symbol $\mathfrak{R}$. The basic expression, called *fuzzy constraint*, in the system is in the form of $\langle\boldsymbol{w}: \varphi\, rel\, \varepsilon\rangle$, where φ is a formula in the first-order logic, $\varepsilon\in[0,1]$ and $rel\in\{<,\leq,>,\geq\}$.

Definition 5.2. An interpretation $\mathcal{I}$ of the system contains a model $\langle\mathcal{W}_{\mathcal{I}},\mathcal{R},\mathcal{D}_{\mathcal{I}},\mathcal{I}\rangle$, where for any $\boldsymbol{w}$, $\boldsymbol{w}^{\mathcal{I}}\in\mathcal{W}_{\mathcal{I}}$ is a possible world, $\mathfrak{R}^{\mathcal{I}}=\mathcal{R}$. Also for any $\boldsymbol{w}$ and any formula $\varphi(x_1,...,x_t)$ with free variables $x_1,...,x_t$, $(\varphi^{\mathcal{I}},\boldsymbol{w}^{\mathcal{I}})$ (simply denoted by $\mathcal{I}(\varphi,\boldsymbol{w})$) is a t-place function on $\mathcal{D}_{\mathcal{I}}$ such that the properties (1)~(5) in Definition 3.1 are satisfied. For any two possible world symbols $\boldsymbol{w}$ and $\boldsymbol{w}'$, $\boldsymbol{w}'$ is said to be *accessible from* $\boldsymbol{w}$ if $\boldsymbol{w}^{\mathcal{I}}\,\mathfrak{R}^{\mathcal{I}}\,\boldsymbol{w}'^{\mathcal{I}}$.

Definition 5.3. A fuzzy constraint $\langle\boldsymbol{w}: \varphi\, rel\, \varepsilon\rangle$, where φ is a formula with free variables $x_1,...,x_t$, is said to be *satisfiable in* $\mathcal{I}$ if there exists a valuation v such that $\mathcal{I}(\varphi,\boldsymbol{w})(v(x_1),...,v(x_t))\, rel\, \varepsilon$. A set of fuzzy constraint S is said to be uniformly satisfiable in an interpretation $\mathcal{I}$ if there exists a valuation v such that every fuzzy constraint of S is satisfiable in $\mathcal{I}$ with respect to v.

The basic properties about the satisfiability of the fuzzy constraints has been discussed in Ref.[13], where some propositions have been listed.

Proposition 5.1. If S is satisfiable in an interpretation $\mathcal{J}$ and $\langle \boldsymbol{w}: \neg\ \varphi\ rel\ n\rangle \in S$, then $S \cup \{\langle \boldsymbol{w}: \varphi\ rel^* 1-\varepsilon \rangle\}$ is satisfiable in $\mathcal{J}$, where $rel \in \{\geq, \leq, >, <\}$ and rel^* is the converse of rel.

Proposition 5.2. If S is satisfiable in $\mathcal{J}$ and $\langle \boldsymbol{w}: \varphi \supset \psi \leq \varepsilon \rangle \in S$, then $S \cup \{\langle \mathbf{w}: \varphi \geq 1-\varepsilon \rangle, \langle \mathbf{w}: \psi \leq \varepsilon \rangle\}$ is satisfiable in $\mathcal{J}$.

Proposition 5.3. If S is satisfiable in $\mathcal{J}$ and $\langle \boldsymbol{w}: \varphi \supset \psi \geq \varepsilon \rangle \in S$, then at least one of the sets $S \cup \{\langle \boldsymbol{w}: \varphi \leq 1-\varepsilon \rangle\}$ and $S \cup \{\langle \mathbf{w}: \psi \geq \varepsilon \rangle\}$ is satisfiable in $\mathcal{J}$.

Proposition 5.4. Let S be a set of fuzzy constraints and $\langle \boldsymbol{w}: \Box \varphi \geq \varepsilon \rangle \in S$. If S is satisfiable in $\mathcal{J}$ then $S \cup \{\langle \mathbf{w}': \varphi \geq \varepsilon \rangle\}$ is satisfiable in $\mathcal{J}$ for any $\boldsymbol{w}'$ such that $\boldsymbol{w}'$ is accessible from $\boldsymbol{w}$.

Remark 2: Proposition 5.4 is also correct if the relation $\geq$ is replaced by $>$. When the constraint of the form $\langle \boldsymbol{w}: \Box \varphi \leq \varepsilon\rangle$ is considered, the condition of the associated proposition should be modified slightly. This is due to the simple fact that $\inf S \leq \varepsilon$ does not necessarily imply the existence of an element in S less than or equal to ε.

Proposition 5.5. Let S be a set of fuzzy constraints and $\langle \boldsymbol{w}: \Box \varphi < \varepsilon\rangle \in S$. If S is satisfiable in $\mathcal{J}$ then $S \cup \{\langle \boldsymbol{w}': \varphi < \varepsilon\rangle\}$ is satisfiable in $\mathcal{J}$ for some $\boldsymbol{w}'$ such that $\boldsymbol{w}'$ is accessible from $\mathbf{w}$. Moreover, if $\langle \boldsymbol{w}: \Box \varphi \leq \varepsilon \rangle \in S$ and S is satisfiable in $\mathcal{J}$ with the condition such that $\mathcal{W}_{\mathcal{J}}$ is finite, then $S \cup \{\langle \boldsymbol{w}': \varphi \leq \varepsilon \rangle\}$ is satisfiable in $\mathcal{J}$ for some $\boldsymbol{w}'$ such that $\boldsymbol{w}'$ is accessible from $\boldsymbol{w}$.

Proposition 5.6. If S is satisfiable in $\mathcal{J}$ and $\langle \boldsymbol{w}: \forall x\varphi(x) \geq \varepsilon\rangle \in S$, then $S \cup \{\langle \boldsymbol{w}: \varphi(y) \geq \varepsilon \rangle\}$ is satisfiable in $\mathcal{J}$, where y is any variable free for x in $\varphi(x)$.

Remark 3: Proposition 5.6 also holds if the relation $\geq$ is replaced by $>$. For a constraint of the form $\langle \boldsymbol{w}: \forall x\varphi(x) \leq \varepsilon \rangle$, the condition in the corresponding proposition should also be modified slightly the same as that in Proposition 5.5. But in this case, the interpretation domain $\mathcal{D}_{\mathcal{J}}$ will be considered.

Proposition 5.7. If S is satisfiable in $\mathcal{J}$ and $\langle \boldsymbol{w}: \forall x\varphi(x) < \varepsilon\rangle \in S$, where x dose not occur freely in any formula of the constraint in S, then $S \cup \{\langle \boldsymbol{w}: \varphi(x) < \varepsilon\rangle\}$ is satisfiable in $\mathcal{J}$. Moreover, if $\langle \boldsymbol{w}: \forall x\varphi(x) \leq \varepsilon\rangle \in S$, where x does not occur freely in any formula of the constraint in S, and S is satisfiable in $\mathcal{J}$ with the condition that $\mathcal{D}_{\mathcal{J}}$ is finite, then $S \cup \{\langle \mathbf{w}: \varphi(x) \leq \varepsilon\rangle\}$ is satisfiable in $\mathcal{J}$.

By Proposition 5.1~5.7, we introduce the following as the basic rules for our reasoning system:

• The reasoning rules about $\mathfrak{R}$:

$(\mathfrak{R}_r)$ $\quad \phi \Rightarrow \langle\langle \boldsymbol{w},\boldsymbol{w}\rangle : \mathfrak{R} \geq 1\rangle$;

$(\mathfrak{R}_s)$ $\quad \langle\langle \boldsymbol{w},\boldsymbol{w}'\rangle : \mathfrak{R} \geq 1\rangle \Rightarrow \langle\langle \boldsymbol{w}',\boldsymbol{w}\rangle : \mathfrak{R} \geq 1\rangle$;

$(\mathfrak{R}_t)$ $\quad \langle\langle \boldsymbol{w},\boldsymbol{w}'\rangle : \mathfrak{R} \geq 1\rangle, \langle\langle \boldsymbol{w}',\boldsymbol{w}''\rangle : \mathfrak{R} \geq 1\rangle \Rightarrow \langle\langle \boldsymbol{w},\boldsymbol{w}''\rangle : \mathfrak{R} \geq 1\rangle$.

• The basic reasoning rules:

$(\neg_{rel})$ $\quad \langle \boldsymbol{w}: \neg\varphi\ rel\ \varepsilon \rangle \Rightarrow \langle \boldsymbol{w}: \varphi\ rel^* 1-\varepsilon \rangle$, where $rel \in \{\geq, \leq, >, <\}$;

$(\supset_{\geq})$ $\quad \langle \boldsymbol{w}: \varphi \supset \psi \geq \varepsilon \rangle \Rightarrow \langle \boldsymbol{w}: \varphi \leq 1-\varepsilon \rangle$ or $\langle \boldsymbol{w}: \psi \geq \varepsilon\rangle$;

$(\supset_{\leq})$ $\quad \langle \boldsymbol{w}: \varphi \supset \psi \leq \varepsilon \rangle \Rightarrow \langle \boldsymbol{w}: \varphi \geq 1-\varepsilon \rangle$ and $\langle \boldsymbol{w}: \psi \leq \varepsilon\rangle$;

$(\Box_{\geq})$ $\quad \langle \boldsymbol{w}: \Box \varphi \geq \varepsilon \rangle, \langle\langle \boldsymbol{w}',\boldsymbol{w}\rangle : \mathfrak{R} \geq 1\rangle \Rightarrow \langle \boldsymbol{w}': \varphi \geq \varepsilon\rangle$;

$(\Box_{<})$ $\quad \langle \boldsymbol{w}: \Box \varphi < \varepsilon \rangle \Rightarrow \langle\langle \boldsymbol{w}',\boldsymbol{w}\rangle : \mathfrak{R} \geq 1\rangle$ and $\langle \boldsymbol{w}': \varphi < \varepsilon\rangle$;

$(\forall_{\geq})$ $\quad \langle \boldsymbol{w}: \forall \varphi(x) \geq \varepsilon \rangle \Rightarrow \langle \boldsymbol{w}: \varphi(y) \geq \varepsilon\rangle$, where y is free for x in $\varphi(x)$;

$(\forall_{<})$ $\quad \langle \boldsymbol{w}: \forall x\varphi(x) < \varepsilon \rangle \Rightarrow \langle \boldsymbol{w}: \varphi(x) < \varepsilon\rangle$;

Remark 4: (a) The reasoning rules about $\mathfrak{R}$ based on the assumption that the relation $\mathcal{R}$ considered in the semantics of the fuzzy first-order modal logic is an equivalent relation on the set of the possible worlds. (b) There are some additional basic reasoning rules can be defined. For instance, the reasoning rules $(\supset_{<})$, $(\supset)$, $(\Box_{>})$ and $(\forall_{>})$ etc. We can define even more reasoning rules, such as $(\wedge_{rel})$, $(\vee_{rel})$, $(\Diamond_{rel})$ etc., where $rel \in \{\leq, \geq, <, >\}$, as an extension of the basic reasoning rules. (c) We may have the reasoning rules $\Box_{\leqslant}$ and $\forall_{\leq}$ which are constrained by the finite semantics in which either the set of possible worlds is finite or the individual domain is finite. However, the condition does not affect the reasoning in the real world since the information used in a real application is usually finite.

6 Conclusion and Further Work

In this paper, we analyze the properties of the fuzzy first-order modal logic, and introduce a formal system of fuzzy reasoning based on the first-order modal logic, and examine the properties about the satisfiability of the reasoning process. The role of studying the fuzzy reasoning is that we can transform the question of deciding whether $\widetilde{\Sigma}\,|\!\approx\langle\varphi,\varepsilon\rangle$ or not into a process to decide if a set of the fuzzy constraints is satisfiable, which has been completely achieved in the fuzzy propositional modal logic. Our further work is to discuss the soundness and of completeness the fuzzy reasoning system and to build a reasoning mechanism which can be used to decide whether or not $\widetilde{\Sigma}\,|\!\approx\langle\varphi,\varepsilon\rangle$ efficiently.

References:

[1] Melvin F, Richard LM. First-Order Modal Logic. Kluwer Academic Publishers, 1998.

[2] Gabbay DM, Hogger CJ, Robinson JA. Handbook of Logic in Artificial Intelligence and Logic Programming. Vol.1-4, Clarendon Press-Oxford, 1994.

[3] Abramsky S, Gabbay DM, Maibaum TSE. Handbook of Logic in Computer Science. Vol.1-3, Clarendon Press-Oxford, 1992.

[4] Orłowska E. Kripke semantics for knowledge representation logics. Studia Logica, 1990,XLIX:255−272.

[5] Fagin RF, Halpern JY, Moses Y, Vardi MY. Reasoning about Knowledge. MIT Press, 1996.

[6] Alefeld G, Herzberger J. Introduction to Interval Computations. New York: Academic Press, 1983.

[7] Dubois D, Prade H. Possibility Theory: An Approach to Computerized Processing of Uncertainty. New York: Plenum Press, 1988.

[8] Hájek P, Harmancová D. A many-valued modal logic. In: Proc. of the IPMU'96. 1996. 1021−1024.

[9] Rodríguez R, Garcia P, Godo L. Using fuzzy similarity relations to revise and update a knowledge base. Mathware and Soft Computing, 1996,3(3):357−370.

[10] Godo L, Rodríguez R. Graded similarity based semantics for nonmonotonic inference. Annals of Mathematics and Artificial Intelligence, 2002,34(1-3):89−105.

[11] Zhang ZY, Sui YF, Cao CG. Formal reasoning system based on fuzzy propositional modal logic. Journal of Software, 2005,16(8):1359−1365 (in English with Chinese abstract). http://www.jos.org.cn/1000-9825/16/1359.htm

[12] Zhang ZY, Sui YF, Cao CG, Wu GH. A formal fuzzy reasoning system and reasoning mechanism based on propositional modal logic. Theoretical Computer Science, 2006,368(1-2):149−160.

[13] Zhang ZY, Sui YF, Cao CG. Description of fuzzy first-order modal logic based on constant domain semantics. In: Proc. of the 10th Int'l Conf. RSFDGrC2005 (Part 1). 2005. 642−650.

附中文参考文献:

[11] 张再跃,眭跃飞,曹存根.基于模糊命题模态逻辑的形式推理系统.软件学报,2005,16(8):1359−1365.

ZHANG Xiao-Ru was born in 1963. She is an associate professor at the School of Computer Science and Engineering of Jiangsu University of Science and Technology. Her current research areas are computer application education, rough set theory and intelligent information processing.

SUI Yue-Fei was born in 1963. He is a professor at the Institute of Computing Technology, the Chinese Academy of Sciences and a CCF senior member. His current research areas are recursion theory, complexity of computation and rough set theory.

ZHANG Zai-Yue was born in 1961. He is a professor at the School of Computer Sciences and Engineering of Jiangsu University of Sciences and Technology. His current research areas are recursion theory, rough set theory and intelligent information processing.

HUANG Zhi-Sheng is a senior researcher in the Knowledge Representation Group of the AI Department, Vrije University of Amsterdam, Netherlands. His current research areas are AI, logics, multimedia, the Semantic Web and ontology engineering.