基于模糊命题模态逻辑的形式推理系统*

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Formal Reasoning System Based on Fuzzy Propositional Modal Logic

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Abstract: The formal reasoning of the fuzzy propositional modal logic based on plausibility degree is considered, and the description of the associated Kripke semantics is given. The fuzzy constraint is introduced and used as a basic expression, the set of reasoning rules based on fuzzy constraint is given and a formal reasoning system is established, and in which the notation of the satisfiability is introduced. The relationship between the fuzzy reasoning and the satisfiability of the associated fuzzy constraints set is studied, and the soundness and completeness of the fuzzy reasoning based on satisfiability are proved.

Key words: propositional modal logic; fuzzy reasoning; formal system

摘 要: 探讨基于可信度的模糊命题模态逻辑的形式推理,给出相关的模糊 Kripke 语义描述.其研究目的旨在解决基于模态命题逻辑的模糊推理的能行问题.在研究过程与方法上,以完全形式化的方法将模糊模态逻辑语法和语

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义统一在一个形式系统中,以模糊约束作为基本表达式,给出推理规则,建立了相应的模糊推理形式系统,并以形式系统中模糊约束集的可满足性来表示模糊推理的有效性,使模糊推理过程变得容易,为最终在计算机上实现基于模态逻辑的模糊推理打下了一定的基础.主要结论是证明了基于可满足性的模糊推理形式系统的可靠性与完备性.

关键词: 命题模态逻辑;模糊推理;形式系统
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1 Introduction

Modal logic[1] is an important logic branch developed firstly in the category of nonclassical logics, and has been widely used as a formalism for knowledge representation in artificial intelligence and an analysis tool in computer science. Along with the study of modal logics, it has been found that modal logic has a close relationship with many other knowledge representation theories. The most well-known result is the connection of the possible world semantics for the modal epistemic logic S_5 with the approximation space in rough set theory[2], where the system S_5 has been shown to be useful in the analysis of knowledge in various areas. Modal logics, however, as an extension of the classical first order logic, are limited to deal with crisp assertions. More often than not, the assertions encountered in the real world are not precise and thus cannot be treated simply by using yes or not. Fuzzy logic directly deals with the notion of vagueness and imprecision. Therefore, it offers an appealing foundation for the generalization of modal logics in order to deal with some vague assertions. Combining fuzzy logic with modal logics, fuzzy modal logics come into our view. Hájek[3] provides a complete axiomatization of fuzzy S_5 system where the accessibility relation is the universal relation, and Godo and Rodríguez give a complete axiomatic system for the extension of Hájec's logic with another modality corresponding to a fuzzy similarity relation.

In this paper we introduce the notation of fuzzy assertion based on plausibility degree and give out a description of the fuzzy propositional modal logic (FPML). In order to find an efficient procedure that can be used to decide whether a fuzzy assertion is a logical consequence of some existing assertions, we establish a formal fuzzy reasoning system. Our main ideal is formed by combining the constraint propagation method introduced in Ref.[4] with the semantics chart method presented in Ref.[5], the former is usually proposed in the context of description logics, and the latter is used to solve the decidability problem of modal propositional calculus. The paper is organized as follows. In Section 2, we make a quick view to propositional modal logic and give out some terminologies and notations that will be used in the paper. In Section 3, we introduce the notation of fuzzy assertion based on propositional modal logic, and discuss the basic properties of the fuzzy propositional modal logic based on fuzzy Kripke semantics. In Section 4, we establish a formal fuzzy reasoning system based on fuzzy Kripke semantics, study the satisfiability of the fuzzy constraints and prove the soundness and completeness of the fuzzy reasoning based on the satisfiability. In the last, we make a conclusion of the paper and propose the further works.

2 Propositional Modal Logic

In general, propositional modal logic (PML) will have its alphabet of symbols:

- a set of proposition symbols, denoted by $PV=\{p_1,p_2,...\}$,
- the logical symbols, $\sim$ (negation), $\wedge$(and), $\vee$(or), $\rightarrow$(material implication),
- the modal operator symbols, $\square$(necessity operator), $\Diamond$ (possibility operator).

The set of wffs of PML is the smallest set satisfying the following conditions:

- p is a wff, for each $p\in PV$,
- if φ is a wff, then $\sim\varphi$, $\square\varphi$, $\Diamond\varphi$ are wffs,

• if φ and ψ are wffs, then $\varphi\wedge\psi$, $\varphi\vee\psi$, $\varphi\rightarrow\psi$ are wffs.

As usual, we shall use $\sim,\rightarrow$ and $\Box$ as the basic connection words and take $\varphi\wedge\psi$ as an abbreviation for $\sim(\varphi\rightarrow\sim\psi)$, and $\varphi\vee\psi$ for $(\sim\varphi\rightarrow\psi)$, $\Diamond\varphi$ for $\sim\Box\sim\varphi$, and $\varphi\leftrightarrow\psi$ for $(\varphi\rightarrow\psi)\wedge(\psi\rightarrow\varphi)$, $\top$ for any tautology $p\vee\sim p$, and $\perp$ for any contradiction $p\wedge\sim p$. There are various types of propositional modal logic systems such as K-system, D-system, T-system, S4-system, S5-system and so on. In order to lay a stress on the key points, we shall limit our discussing on the propositional modal logic system S5 which contains the following axioms and inference rules:

• Axioms:

A_p1 $(\varphi\rightarrow(\psi\rightarrow\varphi))$

A_p2 $((\varphi\rightarrow(\psi\rightarrow\gamma))\rightarrow((\varphi\rightarrow\psi)\rightarrow(\varphi\rightarrow\gamma)))$

A_p3 $((\sim\varphi\rightarrow\sim\psi)\rightarrow(\psi\rightarrow\varphi))$

K $(\Box(\varphi\rightarrow\psi)\rightarrow(\Box\varphi\rightarrow\Box\psi))$

T $(\Box\varphi\rightarrow\varphi)$

E $(\Diamond\varphi\rightarrow\Box\Diamond\varphi)$

• Inference rules:

N (necessity rule) if $\vdash\varphi$ then $\vdash\Box\varphi$

MP (modus ponens) if $\vdash\varphi\rightarrow\psi$ and $\vdash\varphi$ then $\vdash\psi$

A Kripke model for PML is a triple $M=\langle W,R,V\rangle$, where W is a set of possible worlds, R is a binary relation on W, called an accessibility relation, and $V:W\times PV\rightarrow\{0,1\}$ is a truth assignment evaluating the truth value of each propositional symbol in each possible world. The function V can be extended to all wffs recursively in the following way:

(2.1) $V(w,\sim\varphi)=1-V(w,\varphi)$,

(2.2) $V(w,\varphi\wedge\psi)=\min\{V(w,\varphi),V(w,\psi)\}$,

(2.3) $V(w,\varphi\vee\psi)=\max\{V(w,\varphi),V(w,\psi)\}$,

(2.4) $V(w,\varphi\rightarrow\psi)=\max\{1-V(w,\varphi),V(w,\psi)\}$,

(2.5) $V(w,\Box\varphi)=\inf\{V(u,\varphi):\langle u,w\rangle\in R\}$,

(2.6) $V(w,\Diamond\varphi)=\sup\{V(u,\varphi):\langle u,w\rangle\in R\}$.

If $M=\langle W,R,V\rangle$ is model such that R is an equivalent relation on W then M can be viewed as a semantics of S5-system. Let $M=\langle W,R,V\rangle$ be a model of PML, φ be any wff of PML and $w\in W$ be any possible world. We say that φ is true in the possible world w, denoted by $M\models_w\varphi$, if $V(w,\varphi)=1$; we say that φ is false in w, denoted by $M\not\models_w\varphi$, if $V(w,\varphi)=0$. If there exists a possible world $w\in W$ such that $M\models_w\varphi$ then we say that φ is satisfied by w or φ is satisfiable on M. A wff φ is said to be valid on M, or M-valid, denoted by $M\models_w\varphi$, if for all $w\in$W such that $M\models_w\varphi$.

3 Fuzzy Propositional Modal Logic FPML

Propositional modal logic is based on proposition. It discusses the form of proposition and the relationship between propositions. Any wff φ in PML can be viewed as a proposition. By using of the axioms and the inference rules, new formal propositions can be implied. A proposition φ is either true or false in a possible world. However, in a vague system, we can not simply say that a proposition φ, as in PML, is true or false. To cope with this, we introduce the following notations in which plausibility degree of a proposition is considered:

Definition 3.1. A *fuzzy assertion* in fuzzy PML is a pair $\langle\varphi,n\rangle$, where $\varphi\in$ PML is a proposition and $n\in[0,1]$ expresses the plausibility degree of φ. A fuzzy assertion $\langle\varphi,n\rangle$ is called *atomic assertion* if φ is a proposition symbol.

To built the semantics of PML, we shall follow Kripke's semantics for PML Let $M=\langle W,R,V\rangle$ be a triple, where W is a set of possible worlds, R is a binary relation on W, called an accessibility relation, and now V turns to be a function called plausibility degree function, $V:W\times PV\to[0,1]$, such that for each $p\in PV$, $V(w,p)=n$ for some $n\in[0,1]$, where $V(w,p)=n$ can be abbreviated by $w(p)=n$, means that the possible world w considers that the plausibility degree of proposition p is n. As we do in Section 2, see (2.1)-(2.6), the function V can be extended to any proposition of PML recursively.

Definition 3.2. Let $w\in W$ be a possible world and $\langle\varphi,n\rangle$ be a fuzzy assertion in FPML. We say that w satisfies $\langle\varphi,n\rangle$, denoted by $Sat(w,\langle\varphi,n\rangle)$, if $w(\varphi)\geq n$.

Proposition 3.3. Let $M=\langle W,R,V\rangle$ be a Kripke semantic for FPML, $w\in W$ be a possible world, φ,ψ are wffs of PML and $n\in[0,1]$ be a number. Then

(a) $Sat(w,\langle\sim,\varphi,n\rangle)$ iff $w(\varphi)\leq 1-n$;

(b) $Sat(w,\langle\varphi\wedge\psi,n\rangle)$ iff $Sat(w,\langle\varphi,n\rangle)$ and $Sat(w,\langle\psi,n\rangle)$;

(c) $Sat(w,\langle\varphi\vee\psi,n\rangle)$ iff $Sat(w,\langle\varphi,n\rangle)$ or $Sat(w,\langle\psi,n\rangle)$;

(d) $Sat(w,\langle\varphi\to\psi,n\rangle)$ iff $Sat(w,\langle\sim\varphi,n\rangle)$ or $Sat(w,\langle\psi,n\rangle)$;

(e) $Sat(w,\langle\Box\varphi,n\rangle)$ iff $Sat(u,\langle\varphi,n\rangle)$ for all $u\in W$ such that $\langle u,w\rangle\in R$;

(f) $Sat(w,\langle\Diamond\varphi,n\rangle)$ iff $Sat(u,\langle\varphi,n\rangle)$ for some $u\in W$ such that $\langle u,w\rangle\in R$.

Definition 3.4. Let $M=\langle W,R,V\rangle$ be a model defined as above, and A be a set of fuzzy assertions. If there exists a $w\in W$ such that $Sat(w,\gamma)$ for all $\gamma\in A$ then A is said to be satisfiable in M, and is denoted by $M\models_{\approx w}A$. If for all possible worlds $w\in W$, $M\models_{\approx w}A$ then A is said to be valid in M, and is denoted by $M\models_{\approx}A$.

Proposition 3.5. For any semantics model M of FPML, following properties hold:

(a) $M\models_{\approx}\langle A_p1,0.5\rangle$, (b) $M\models_{\approx}\langle A_p2,0.5\rangle$, (c) $M\models_{\approx}\langle A_p3,0.5\rangle$, (d) $M\models_{\approx}\langle \boldsymbol{K},0.5\rangle$, (e) $M\models_{\approx}\langle \boldsymbol{T},0.5\rangle$, (f) $M\models_{\approx}\langle \boldsymbol{E},0.5\rangle$.

Proposition 3.6. If $M\models_{\approx}\langle\varphi,n\rangle$ then $M\models_{\approx}\langle\Box\varphi,n\rangle$.

Proposition 3.7. If $M\models_{\approx}\langle\varphi\to\psi,n\rangle$ and $M\models_{\approx}\langle\varphi,m\rangle$, where $n,m\in[0,1]$ such that $n>1-m$ then $M\models_{\approx}\langle\psi,n\rangle$.

Proposition 3.7 gives out a sort of modus ponens over assertions, from which we can see that the plausibility degree of ψ depends on the plausibility degrees of $\varphi\to\psi$ and φ. When the plausibility degree of φ is very small, the plausibility degree of ψ can hardly be confirmed.

4 Formal Fuzzy Reasoning Based on PML

Let Σ be a set of wffs of PML. A wff φ is said to be a logical consequence of Σ, denoted by $\Sigma\models\varphi$, if every model of Σ is also a model of φ. In FPML, we have following definitions.

Definition 4.1. A set of fuzzy assertions is called a fuzzy knowledge base. Let $\tilde{\Sigma}$ be a fuzzy knowledge base and $\langle\varphi,n\rangle$ be any fuzzy assertion. If any model of $\tilde{\Sigma}$ is also a model of $\langle\varphi,n\rangle$ then we say that assertion $\langle\varphi,n\rangle$ is a logical consequence of $\tilde{\Sigma}$, which is denoted by $\tilde{\Sigma}\models_{\approx}\langle\varphi,n\rangle$.

For example, by Propositions 3.6 and 3.7, we immediately have that $\{\langle\varphi,n\rangle\}\models_{\approx}\langle\Box\varphi,n\rangle$ and $\{\langle\varphi\to\psi,n\rangle,\langle\varphi,m\rangle\}\models_{\approx}\langle\psi,n\rangle$ for $n>1-$m.

The process of deciding whether $\tilde{\Sigma}\models_{\approx}\gamma$ is called a fuzzy reasoning procedure based on PML. We shall develop a reasoning mechanism about fuzzy assertions in this section.

Definition 4.2. The alphabet of our fuzzy reasoning system contains a set of the symbols used in PML, a set of possible worlds symbols $\mathbf{w}_1,\mathbf{w}_2,\ldots$, a set of relation symbols $\{<,\leq,>,\geq\}$ and a special symbol R. The fuzzy constraint in the fuzzy reasoning system is the expression in the form of $\langle\mathbf{w}:\varphi\ rel\ n\rangle$ or $\langle\langle\mathbf{w},\mathbf{w}'\rangle: R\geq 1\rangle$, where $\varphi\in$PML, $n\in[0,1]$ and $rel\in\{<,\leq,>,\geq\}$.

Definition 4.3. An interpretation I of the system contains a interpretation domain W such that for any $\mathbf{w}$, its interpretation $\mathbf{w}^I\in W$ is a mapping from PV into [0,1], and the interpretation R^I is a relation on W.

Definition 4.4. We say that a fuzzy constraint $\langle\mathbf{w}:\varphi\ rel\ n\rangle$ (resp.$\langle\langle\mathbf{w},\mathbf{w}'\rangle:R\geq 1\rangle$) is satisfiable in an interpretation I if $\mathbf{w}^I(\varphi)\ rel\ n$ (resp.$\langle\mathbf{w}^I,\mathbf{w}'^I\rangle\in R^I$); a set S of fuzzy constraint is satisfiable in I if every element of S is satisfiable in

I. A fuzzy constraint γ (or a set of constraints S) is said to be satisfiable if there exists an interpretation I such that γ (or S) is satisfiable in I.

The system contains the following reasoning rules:

• The reasoning rules about R:

(R_r) $\varnothing \Rightarrow \langle\langle \mathbf{w},\mathbf{w}\rangle: R \geq 1\rangle$;

(R_s) $\langle\langle \mathbf{w},\mathbf{w}'\rangle: R \geq 1\rangle \Rightarrow \langle\langle \mathbf{w}',\mathbf{w}\rangle: R \geq 1\rangle$;

(R_t) $\langle\langle \mathbf{w},\mathbf{w}'\rangle: R \geq 1\rangle,\langle\langle \mathbf{w}',\mathbf{w}''\rangle: R \geq 1\rangle \Rightarrow \langle\langle \mathbf{w},\mathbf{w}''\rangle: R \geq 1\rangle$.

• The basic reasoning rules:

$(\sim\geq)$ $\langle \mathbf{w}:\sim\varphi\geq n\rangle \Rightarrow \langle \mathbf{w}:\varphi\leq 1-n\rangle$;

$(\sim\leq)$ $\langle \mathbf{w}:\sim\varphi\leq n\rangle \Rightarrow \langle \mathbf{w}:\varphi\geq 1-n\rangle$;

$(\rightarrow\geq)$ $\langle \mathbf{w}:\varphi\rightarrow\psi\geq n\rangle \Rightarrow \langle \mathbf{w}:\varphi\leq 1-n\rangle \mid \langle \mathbf{w}:\psi\geq n\rangle$;

$(\rightarrow\leq)$ $\langle \mathbf{w}:\varphi\rightarrow\psi\leq n\rangle \Rightarrow \langle \mathbf{w}:\varphi\geq 1-n\rangle,\langle \mathbf{w}:\psi\leq n\rangle$;

$(\square\geq)$ $\langle \mathbf{w}:\square\varphi\geq n\rangle,\langle\langle \mathbf{w}',\mathbf{w}\rangle:R \geq 1\rangle \Rightarrow \langle \mathbf{w}':\varphi\geq n\rangle$;

$(\square\leq)$ $\langle \mathbf{w}:\square\varphi\leq n\rangle,\langle\langle \mathbf{w}',\mathbf{w}\rangle:R \geq 1\rangle \Rightarrow \langle \mathbf{w}':\varphi\leq n\rangle$.

The rules for the case < and > are quite similar. From basic reasoning rules one can easily define reasoning rules $(\wedge_{rel})$, $(\vee_{rel})$, $(\lozenge_{rel})$, where $rel\in\{\leq,\geq,<,>\}$, as an extension of the basic reasoning rules.

Definition 4.5. Two fuzzy constraints ξ, ζ are said to be a conjugated pair if one of the following conditions holds:

(4.1) $\xi=\langle \mathbf{w}:\varphi\geq n\rangle$, $\zeta=\langle \mathbf{w}:\varphi\leq m\rangle$ and $n>m$;

(4.2) $\xi=\langle \mathbf{w}:\varphi\geq n\rangle$, $\zeta=\langle \mathbf{w}:\varphi< m\rangle$ and $n\geq m$;

(4.3) $\xi=\langle \mathbf{w}:\varphi> n\rangle$, $\zeta=\langle \mathbf{w}:\varphi\leq m\rangle$ and $n\geq m$;

(4.4) $\xi=\langle \mathbf{w}:\varphi> n\rangle$, $\zeta=\langle \mathbf{w}:\varphi< m\rangle$ and $n\geq m$.

Definition 4.6. A set of fuzzy constraints S contains a *clash* if it contains a conjugated pair.

Proposition 4.7. If S is a set of fuzzy constraints and contains a clash then S can not be satisfied in any interpretation I.

Lemma 4.8. If fuzzy constraint $\langle \mathbf{w}:\varphi rel\ n\rangle$ is satisfiable in some interpretation Proposition 4.7. If S is a set of fuzzy constraints and contains a clash then S can not be satisfied in any interpretation I. then there exists a model $M=\langle W,R,V\rangle$ such that $\mathbf{w}^I\in W$ and for each $w\in W$, $w(\varphi)rel\ n$.

Proof. We prove the lemma by induction on the length of the formula φ.

The basic step is quite simple. If φ is a proposition symbol, we define $W=\{\mathbf{w}^I\}$, $R=\{\langle \mathbf{w}^I,\mathbf{w}^I\rangle\}$ and $V(\mathbf{w}^I,p)=\mathbf{w}^I(p)$ for every $p\in PV$, then the model $M=\langle W,R,V\rangle$ is what we need.

Assume that φ is $\sim\psi$. Since $\langle \mathbf{w}:\varphi\ rel\ n\rangle$ is satisfiable in I, $\langle \mathbf{w}:\psi\ rel^*\ 1-n\rangle$ is satisfiable in I, where rel^* is the converse of rel. By induction assumption, we have a model M such that $\mathbf{w}^I\in W$ and for every $\mathbf{w}\in W$, $w(\psi)\ rel^*\ 1-n$. Notice that $w(\psi)\ rel^*\ 1-n$ iff $w(\sim\psi)\ rel\ n$, thus M is also the model we need.

Suppose φ is $\psi_1\rightarrow\psi_2$. There are two cases according to $rel\in\{>,\geq\}$ and $rel\in\{\leq,<\}$. If $rel\in\{>,\geq\}$ then either $\langle \mathbf{w}:\psi_1 rel^*\ 1-n\rangle$ or $\langle \mathbf{w}:\psi_2 rel\ n\rangle$ is satisfiable in I. By induction hypothesis, the model obtained according to either $\langle \mathbf{w}:\psi_1 rel^*\ 1-n\rangle$ or $\langle \mathbf{w}:\psi_2 rel\ n\rangle$ is what we need. If $rel\in\{<,\leq\}$ then both $\langle \mathbf{w}:\psi_1 rel^*\ 1-n\rangle$ and $\langle \mathbf{w}:\psi_2 rel\ n\rangle$ are satisfiable in I. Thus, by induction hypothesis, we have two models, M_1, M_2 say, obtained by the facts that both $\langle \mathbf{w}:\psi_1 rel^*\ 1-n\rangle$ and $\langle \mathbf{w}:\psi_2 rel\ n\rangle$ are satisfiable in I respectively. Since $\mathbf{w}^I$ is in both M_1 and M_2, $W_1\cap W_2\neq\varnothing$. Let $W=W_1\cap W_2$ and $R=R^I\lceil W$. Then the model $M=\langle W,R\rangle$ satisfies the lemma's condition.

Assume that φ is $\square\psi$ and that $\langle \mathbf{w}:\square\psi rel\ n\rangle$ is satisfiable in I. If $rel\in\{>,\geq\}$ then $\langle \mathbf{w}:\psi rel\ n\rangle$ is also satisfiable in I, thus the model we need exists. If $rel\in\{<,\leq\}$ then there exists a symbol $\mathbf{w}_1$ such that $\langle \mathbf{w}_1{}^I,\mathbf{w}^I\rangle\in R^I$ and $\langle \mathbf{w}_1:\psi rel\ n\rangle$ is satisfiable in I. By induction hypothesis, there exists a model M_1 such that $\mathbf{w}_1\in W_1$ and $w(\psi)rel\ n$ for any $w\in W_1$.

Let $W=W_1\cup\{\mathbf{w}^I\}$, $R=R^I\lceil W$. It is easy to verify that M is the model we need. □

Corollary 4.9. Let $S=\{\langle \mathbf{w}:\varphi_i rel_i\ n_i\rangle:1\le i\le m\}$ be a set of fuzzy constraints. If S is satisfiable then there exists a model M such that the interpretation of $\mathbf{w}$ is in W and $w(\varphi_i)\ rel_i\ n_i$ for each $w\in W$ and each $\langle \mathbf{w}:\varphi_i rel_i\ n_i\rangle\in S$.

We may reduce the fuzzy reasoning problem to the satisfiability of a set of fuzzy constraints. The following theorems show that our reasoning mechanism based on satisfiability is sound and complete. To decide whether $\tilde{\Sigma}\mid\!\approx\langle\varphi,n\rangle$, let $S_{\tilde{\Sigma}}=\{\langle \mathbf{w}:\psi\ge n_\psi\rangle:\langle\psi,n_\psi\rangle\in\tilde{\Sigma}\}$, it then follows that

Theorem 4.10. $\tilde{\Sigma}\mid\!\approx\langle\varphi,n\rangle$ iff $S_{\tilde{\Sigma}}\cup\{\langle \mathbf{w}:\varphi<n\rangle\}$ is not satisfiable.

Proof. If $S_{\tilde{\Sigma}}\cup\{\langle \mathbf{w}:\varphi<n\rangle\}$ is satisfiable in some I, then by Corollary 4.9 there exists a model M such that $\mathbf{w}^I\in M$. M is obviously a model of $\tilde{\Sigma}$, but not a model of $\langle\varphi,n\rangle$, this is because that for all $w\in M$, $w(\psi)\ge n_\psi$ for any $\langle\psi,n_\psi\rangle\in\tilde{\Sigma}$ and $\mathbf{w}^I(\varphi)<n$, thus $\tilde{\Sigma}\mid\!\approx\langle\varphi,n\rangle$. Conversely, if $\tilde{\Sigma}\mid\!\approx\langle\varphi,n\rangle$, then there exists a model $M=\langle W,R,V\rangle$, and a possible world $w\in W$ such that $w(\psi)\ge m$ for any $\langle\psi,m\rangle\in\tilde{\Sigma}$ and $w(\varphi)<n$. Let I be an interpretation such that $\mathbf{w}^I=w$. Then $S_{\tilde{\Sigma}}\cup\{\langle \mathbf{w}:\varphi<n\rangle\}$ is satisfied by interpretation I.

Following example shows how our reasoning works.

Example 1. To decide whether $\{\langle\Diamond\varphi,0.7\rangle,\langle\Box\psi,0.6\rangle\}\mid\!\approx\langle\Diamond(\varphi\wedge\psi),0.6\rangle$ or not?

Let $S_{\tilde{\Sigma}}=\{\langle \mathbf{w}:\Diamond\varphi\ge0.7\rangle,\langle \mathbf{w}:\Box\psi\ge0.6\rangle\}$ and $S=S_{\tilde{\Sigma}}\cup\langle \mathbf{w}:\Diamond(\varphi\wedge\psi)<0.6\rangle$. The reasoning procedure is as follows:

(1) $\langle \mathbf{w}:\Diamond\varphi\ge0.7\rangle$	Hypothesis
(2) $\langle \mathbf{w}:\Box\psi\ge0.6\rangle$	Hypothesis
(3) $\langle \mathbf{w}:\Diamond(\varphi\wedge\psi)<0.6\rangle$	Hypothesis
(4) $\langle\langle \mathbf{w}',\mathbf{w}\rangle:R\ge1\rangle,\langle \mathbf{w}':\varphi\ge0.7\rangle$	(1)($\Diamond\ge$)
(5) $\langle \mathbf{w}':\psi\ge0.6\rangle$	(2)(4)($\Box\ge$)
(6) $\langle \mathbf{w}':\varphi\wedge\psi<0.6\rangle$	(3)(4)($\Diamond<$)
(7) $\langle \mathbf{w}':\varphi<0.6\rangle\mid\langle \mathbf{w}':\psi<0.6\rangle$	(6)($\wedge<$)

We have $S=\{(1),(2),(3)\}$ at the beginning of our reasoning, then $S_1=S\cup\{(4)\}$, then $S_2=S_1\cup\{(5)\}$, then $S_3=S_2\cup\{(6)\}$. There are two educed sets S' and S'' of S_3, where $S'=S_3\cup\{\langle \mathbf{w}':\varphi<0.6\rangle\}$ and $S''=S_3\cup\{\langle \mathbf{w}':\psi<0.6\rangle\}$. This is because the constraint (7) is obtained by using $(\wedge_<)$ on (6). Since both S' and S'' contain a clash, by Proposition 4.7 both S' and S'' can not be satisfied in any interpretation I, thus the set $S=S_{\tilde{\Sigma}}\cup\langle \mathbf{w}:\Diamond(\varphi\wedge\psi)<0.6\rangle$ can not be satisfied by any interpretation I. By Theorem 4.10, we know that $\langle\Diamond(\varphi\wedge\psi),0.6\rangle$ is a logical consequence of $\{\langle\Diamond\varphi,0.7\rangle,\langle\Box\psi,0.6\rangle\}$.

5 Conclusion and Further Work

In this paper we have introduced a formal fuzzy reasoning system based on propositional modal logic and established the relationship between the reasoning procedure $\tilde{\Sigma}\mid\!\approx\langle\varphi,n\rangle$ and the satisfiability of some set of fuzzy constraints. Compared with the works that have been done by many logical researchers, the distinctive feature of our work is that the formal fuzzy reasoning system established in this paper is not for the fuzzy propositional modal logic itself, but for the reasoning based on the fuzzy propositional modal logic. Fuzzy constraints in our system contain semantic information, thus to decide if a set of fuzzy constraints is satisfiable is more practicable than to verify that every model of $\tilde{\Sigma}$ is the model of $\langle\varphi,n\rangle$. Our intention is to offer a method which could be used to realize the reasoning based on the modal logic in a computer eventually. However, there are many technical works that have to be done before we reach our goal. Our further work is to build an efficient reasoning mechanism based on fuzzy constraints, which could be used to realize in computer to decide whether a fuzzy assertion is a logical consequence of some existing assertions. The basic idea of our further work is to construct a it reasoning tree in which every branch is a set of the constraints reduced from the original one during our reasoning procedure, and convert the reasoning problem into the satisfiability problem of these constraint sets.

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