空间曲线几何 Hermite 插值的 B 样条方法*

朱春钢+, 王仁宏

(大连理工大学 数学科学研究所,辽宁 大连 116024)

Geometric Hermite Interpolation for Space Curves by B-Spline

ZHU Chun-Gang+, WANG Ren-Hong

(Institute of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China)

+ Corresponding author: Phn: +86-411-84708351 ext 8407, E-mail: zhuchungang@hotmail.com

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Abstract: This paper considers the space GC^2 Hermite interpolation by cubic B-spline curve which is based on de Boor's idea for constructing the planar GC^2 Hermite interpolation. In addition to position and tangent direction, the curvature vector is interpolated at each point. It is proved that under appropriate assumptions the interpolant exists locally with two degrees of freedom and the 4th order accuracy.

Key words: spline; curve; interpolation; geometric smoothness; accuracy

摘 要: 在给定的 GC^2 插值条件,利用 de Boor 的构造平面曲线的 GC^2 -Hermite 插值方法,构造了一条具有两个自由度的三次 B 样条插值曲线,并证明插值曲线是局部存在的且具有 4 阶精度.

关键词: 样条;曲线;插值;几何光滑;精度

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1 Introduction

Curve interpolation is one of fundamental issues in CAGD. It is well known that the classical Hermite interpolation can obtain a parametric curve with a higher parametric smoothness. The construction of twice continuously differentiable cubic spline interpolants usually involves the solution of a global system of equations[1]. Since the geometric continuity is generally weaker than the parametric continuity, it is expected to drop down the degree of the interpolant. Thus the geometric Hermite interpolation (GHI) was introduced and studied[2−6]. The GHI is the Hermite interpolation based on geometric continuity. In particular, the aim of the quadratic geometric(GC^2)

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ZHU Chun-Gang was born in 1977. He is a Ph.D. candidate at the Institute of Mathematical Sciences, Dalian University of Technology. His current research areas are CAGD and computational geometry. **WANG Ren-Hong** was born in 1937. He is a professor and doctoral supervisor at the Institute of Mathematical Sciences, Dalian University of Technology. His research areas are computational geometry, CAGD, multivariate splines and approximation theory.

Hermite interpolation is to find a curve which interpolates positions, tangent directors, and curvatures at the endpoints. The quintic polynomials are needed for C^2-Hermite interpolation. However, the following researches show that the degree of GC^2-Hermite interpolants can be dropped down, and the approximation order will be good. de Boor *et al*.[2] showed that if the curvature at one endpoint is not vanished, then the interpolant of a planar curve exists locally and the approximation order is 6. Höllig and Koch[3] considered the approximation of space curve by the cubic Bézier curve, and showed that if the torsion at one endpoint is not vanished, then the solution exists locally, and the approximation order is 5. Furthermore, they also considered[4] the planar GC^2 Hermite interpolation by quadratic B-splines. Xu and Shi[5] also considered this question by space quartic Bézier curve, and showed that if the torsion at one endpoint is not vanished, then there exists $H>0$ such that for $0<h<H$, the space GHI problem has solutions with one degree of freedom and approximation order 6. Recent implementations show that the GHI performs excellent in various applications. Unfortunately, most of the researches are mainly for planar curves. Höllig and Koch[3] didn't consider curvature vector, so their interpolation is not the GC^2 Hermite interpolation. In engineering and mechanics, a better interpolant has lots of better properties, such as smaller energy, smaller arc length, etc. For space GC^2 Hermite interpolation, cubic polynomials are not enough because the curvatures become vectors in R^3. This paper presents a new scheme for space GC^2 Hermite interpolation by cubic B-spline curve. The advantages of the new scheme over others are not only dropping down the degree of the interpolant but also possessing two degrees of freedom to control the shape-forming of the interpolant to satisfy various requirements.

Our main result is: If $\mathbf{r}=\mathbf{r}(s), s\in[0,l]$ is a smooth curve (C^5 is enough) with nonvanishing torsion at one endpoint, then there exists $H>0$ such that the space GHI problem is solvable for $0<h<H$. Moreover, the solution possesses two degrees of freedom, and approximation order is 4.

The paper is organized as follows. The second section discusses the construction of the interpolant. The third section proves the local existence of the interpolant, and the approximation order is considered in section 4. Section 5 gives some examples. Finally, Section 6 presents the conclusions and future work.

2 Construction of the Interpolant

The space GHI conditions[4] can be represented as follows:

$$\mathbf{r}_i=\mathbf{b}(i),\mathbf{d}_i=\frac{\mathbf{b}'(i)}{|\mathbf{b}'(i)|},\mathbf{k}_i=\frac{\mathbf{b}'(i)\times\mathbf{b}''(i)}{|\mathbf{b}'(i)|^3},i=0,1, \tag{1}$$

where $\mathrm{r}_i,\mathrm{d}_i,\mathrm{k}_i$ are the given endpoint positions(without loss of generality, we assume $\mathrm{r}_0=\mathrm{r}(0)=0,\mathrm{r}_1=\mathrm{r}(h),h\in(0,l]$), tangent directions and normal curvature vectors on the curve $\mathrm{r}=\mathrm{r}(s), s\in[0,l]$. We represent $\mathbf{b}(t)$ as a cubic B-spline curve

$$\mathbf{b}(t)=\sum_{i=0}^{4}\mathbf{b}_i N_{i,3}(t),t\in[0,1], \tag{2}$$

where $\{\mathbf{b}_i\}$ are the control points, and $\{N_{i,3}(t)\}$ are the B-spline basis functions defined on the knot vector

$$\mathbf{U}=\{0,0,0,0,z,1,1,1,1\},z\in(0,1).$$

The first two conditions in (1) imply

$$\mathbf{b}_0=\mathbf{b}(0)=\mathbf{0},\mathbf{b}_4=\mathbf{b}(1)=\mathbf{r}_1, \tag{3}$$

$$\Delta\mathbf{b}_0=l_0\mathbf{d}_0,\Delta\mathbf{b}_3=l_1\mathbf{d}_1,l_0,l_1>0. \tag{4}$$

Hence from the third condition in (1), we have

$$\mathbf{b}_2=u_0\mathbf{d}_0+\frac{3l_0^{\ 2}}{2z}\mathbf{k}_0\times\mathbf{d}_0,u_0\in R, \tag{5}$$

$$\mathbf{b}_2 = \mathbf{b}_4 - u_1\mathbf{d}_1 + \frac{3l_1^{\ 2}}{2(1-z)}\mathbf{k}_1 \times \mathbf{d}_1, u_1 \in R. \tag{6}$$

Thus we obtain the solvable condition of the space GHI problem

$$(u_0 + l_0)\mathbf{d}_0 + (u_1 + l_1)\mathbf{d}_1 = \mathbf{b}_4 - \frac{3l_0^{\ 2}}{2z}\mathbf{k}_0 \times \mathbf{d}_0 + \frac{3l_1^{\ 2}}{2(1-z)}\mathbf{k}_1 \times \mathbf{d}_1. \tag{7}$$

In fact, if u_0, u_1, l_0, and l_1 satisfy (7) and $l_0 > 0, l_1 > 0$, then (3), (4), and (5) or (6) give the solutions of the GHI problem. We discuss the problem in terms of the value of $\mathbf{k}_0 \times \mathbf{k}_1$.

2.1 $\mathbf{k}_0 \times \mathbf{k}_1 = \mathbf{0}$

2.1.1 $\mathbf{b}_4 \bullet \mathbf{k}_1 \neq 0$

Let (7) make inner products respectively with $\mathbf{k}_0, \mathbf{k}_1$, we get $\mathbf{b}_4 \bullet \mathbf{k}_1 = 0$. Then the space GHI problem is unsolvable. However, the Corollary in section 3 indicates that we can subdivide the problem and find a piecewise cubic B-spline curve satisfying the GC^2-condition for any smooth curve $\mathbf{r} = \mathbf{r}(s), s \in [0, l]$ with nonvanishing torsion anywhere.

2.1.2 $\mathbf{b}_4 \bullet \mathbf{k}_1 = 0$

This case can be dealt with by planar cubic Bézier curves or quadratic B-spline curves[2,6].

2.2 $\mathbf{k}_0 \times \mathbf{k}_1 \neq \mathbf{0}$

2.2.1 $\mathbf{d}_0 \bullet \mathbf{k}_1 = \mathbf{d}_1 \bullet \mathbf{k}_0 = 0$

Let (7) make inner products respectively with $\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_0 \times \mathbf{k}_1$, we get

$$l_0^{\ 2} = -\frac{2z}{3}\frac{\mathbf{b}_4 \bullet \mathbf{k}_1}{(\mathbf{k}_0, \mathbf{k}_1, \mathbf{d}_0)}, \tag{8}$$

$$l_1^{\ 2} = -\frac{2(1-z)\mathbf{b}_4 \bullet \mathbf{k}_0}{3(\mathbf{k}_0, \mathbf{k}_1, \mathbf{d}_1)}, \tag{9}$$

$$u_0(\mathbf{k}_0, \mathbf{k}_1, \mathbf{d}_0) + u_1(\mathbf{k}_0, \mathbf{k}_1, \mathbf{d}_1) = (\mathbf{k}_0, \mathbf{k}_1, \mathbf{b}_4 - l_0\mathbf{d}_0 - l_1\mathbf{d}_1). \tag{10}$$

This proves the following result.

Theorem 1. Suppose that $\mathbf{k}_0 \times \mathbf{k}_1 \neq \mathbf{0}$ and $\mathbf{d}_0 \bullet \mathbf{k}_1 = \mathbf{d}_1 \bullet \mathbf{k}_0 = 0$, then the space GHI problem is solvable if and only if

$$(\mathbf{b}_4 \bullet \mathbf{k}_1)(\mathbf{k}_0, \mathbf{k}_1, \mathbf{d}_0) < 0, (\mathbf{b}_4 \bullet \mathbf{k}_0)(\mathbf{k}_0, \mathbf{k}_1, \mathbf{d}_1) < 0.$$

In this case, (l_0, l_1) is determined by (8),(9) and (u_0, u_1) varies in the straight line given by (10). Furthermore, (u_0, u_1) 's variations in the line (10) and z 's variations in (0,1) can be viewed as two shape parameters to control the shape-forming of the B-spline interpolation curve.

2.2.2 $\mathbf{d}_0 \bullet \mathbf{k}_1 \neq 0$ or $\mathbf{d}_1 \bullet \mathbf{k}_0 \neq 0$

Let L_0, L_1 and π_0, π_1 denote the tangents and the osculating planes at the endpoints respectively. They can be represented as follows:

$$\begin{aligned} &\pi_0 : \mathbf{x} \bullet \mathbf{k}_0 = 0, \\ &\pi_1 : (\mathbf{x} - \mathbf{b}_4) \bullet \mathbf{k}_1 = 0, \\ &L_0 : \mathbf{x} = m\mathbf{d}_0, m \in R, \\ &L_1 : \mathbf{x} - \mathbf{b}_4 = m\mathbf{d}_1, m \in R. \end{aligned}$$

In this case, we have $\mathbf{k}_0 \times \mathbf{k}_1 \neq \mathbf{0}$, i.e. $\pi_0 \cap \pi_1 \neq \phi$. Denote by L the intersection line of π_0 and π_1. It is clear that the direction of L is $\mathbf{k}_0 \times \mathbf{k}_1$.

First, we consider the case of $\mathbf{d}_0 \bullet \mathbf{k}_1 \neq 0$, i.e. L_0 is unparallel with L. Denote by $\mathbf{q}$ the intersection point of L_0 and L. Note that $\mathbf{q} \in L_0, \mathbf{q} \in \pi_1$, we get

$$\mathbf{q} = c_0\mathbf{d}_0, (\mathbf{q} - \mathbf{b}_4) \bullet \mathbf{k}_1 = 0.$$

This leads to

$$c_0 = \frac{\mathbf{b}_4 \bullet \mathbf{k}_1}{\mathbf{d}_0 \bullet \mathbf{k}_1}.$$

Therefore, for checking Eqs.(5), (6), they are sufficient to show that

$$\mathbf{b}_2 = c_0\mathbf{d}_0 + c\mathbf{k}_0 \times \mathbf{k}_1, c \in R. \tag{11}$$

From (1), (3), and (4), we have

$$l_0^{\,2} = \frac{2z}{3}c(\mathbf{k}_1 \bullet \mathbf{d}_0), \tag{12}$$

$$l_1^{\,2} = \frac{2(1-z)\left[(\mathbf{b}_4 - c_0\mathbf{d}_0, \mathbf{d}_1, \mathbf{k}_1) - c(\mathbf{k}_0 \bullet \mathbf{d}_1)|\mathbf{k}_1|^2\right]}{3|\mathbf{k}_1|^2}. \tag{13}$$

Thus we get:

Theorem 2. If $\mathbf{k}_0 \times \mathbf{k}_1 \neq \mathbf{0}$ and $\mathbf{d}_0 \bullet \mathbf{k}_1 \neq 0$, then the solution to the space GHI problem is given by (11)–(13), where

$$c \in C := \left\{c \in R : c(\mathbf{k}_1 \bullet \mathbf{d}_0) > 0, (\mathbf{b}_4 - c_0\mathbf{d}_0, \mathbf{d}_1, \mathbf{k}_1) - c(\mathbf{k}_0 \bullet \mathbf{d}_1)|\mathbf{k}_1|^2 > 0\right\}.$$

Moreover, c's variations in C and z's variations in (0,1) can be viewed as two shape parameters to control the shape-forming of the B-spline interpolation curve.

Similarly we can discuss the case of $\mathbf{d}_1 \bullet \mathbf{k}_0 \neq 0$.

3 Existence of the Interpolant

Suppose that $\mathbf{r} = \mathbf{r}(s)$ is a smooth curve, $s \in [0,l]$ as an arc length parameter. The conditions in (1) imply

$$\mathbf{b}_0 = \mathbf{r}(0) = \mathbf{0},\ \mathbf{d}_0 = \mathbf{r}'(0),\ \mathbf{k}_0 = \mathbf{k}(0), \tag{14}$$

$$\mathbf{b}_4 = \mathbf{r}(h),\ \mathbf{d}_1 = \mathbf{r}'(h),\ \mathbf{k}_1 = \mathbf{k}(h),\ h \in (0,l], \tag{15}$$

where $\mathbf{k}(s)$ is the curvature vector of $\mathbf{r}(s)$. The interpolant $\mathbf{b}(t)$ can be regarded as an approximation of the original curve $\mathbf{r} = \mathbf{r}(s)$, $s \in [0,h]$, if h is sufficiently small. The aim of this section is to prove the local existence of the space GHI problem.

We expand the curve $\mathbf{r} = \mathbf{r}(s)$ at $s = 0$

$$\mathbf{r}(s) = \mathbf{r}'s + \frac{1}{2!}\mathbf{r}''s^2 + \frac{1}{3!}\mathbf{r}'''s^3 + O(s^4), \tag{16}$$

where $\mathbf{r}^{(i)} = \mathbf{r}^{(i)}(0), i = 1,2,3$. This follows

$$\mathbf{k}(s) = \mathbf{r}'(s) \times \mathbf{r}''(s) = \mathbf{r}' \times \mathbf{r}'' + \mathbf{r}' \times \mathbf{r}'''s + \frac{1}{2!}(\mathbf{r}'' \times \mathbf{r}''' + \mathbf{r}' \times \mathbf{r}^{(4)})s^2 + O(s^3). \tag{17}$$

From (14)–(17), we find

$$\mathbf{d}_0 \bullet \mathbf{k}_1 = \frac{1}{2}(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')h^2 + O(h^3),$$

$$\mathbf{d}_1 \bullet \mathbf{k}_0 = \frac{1}{2}(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')h^2 + O(h^3),$$

$$\mathbf{b}_4 \bullet \mathbf{k}_1 = \frac{1}{6}(\mathbf{r}', \mathbf{r}'', \mathbf{r}''')h^3 + O(h^4),$$

$$\mathbf{k}_0 \times \mathbf{k}_1 = (\mathbf{r}', \mathbf{r}'', \mathbf{r}''')h + O(h^2).$$

If $(\mathbf{r}',\mathbf{r}'',\mathbf{r}''')\neq 0$, we get $\mathbf{k}_0\times\mathbf{k}_1\neq\mathbf{0}$, $\mathbf{d}_0\bullet\mathbf{k}_1\neq 0$ and $|\mathbf{k}_0|\neq 0$. Therefore, this case can be dealt with by Theorem 2. Furthermore, we obtain

$$c_0=\frac{\mathbf{b}_4\bullet\mathbf{k}_1}{\mathbf{d}_0\bullet\mathbf{k}_1}=\frac{1}{3}h+O(h^2),$$

$$(\mathbf{b}_4-c_0\mathbf{d}_0,\mathbf{d}_1,\mathbf{k}_1)=\frac{1}{6}|\mathbf{k}_0|^2h^2+O(h^3).$$

(1) If $(\mathbf{r}',\mathbf{r}'',\mathbf{r}''')>0$ and h is sufficiently small, then $\mathbf{k}_0\bullet\mathbf{d}_1>0$, $\mathbf{k}_1\bullet\mathbf{d}_0>0$, we have

$$c\in C:=\left\{c\in R:0<c<\frac{(\mathbf{b}_4-c_0\mathbf{d}_0,\mathbf{d}_1,\mathbf{k}_1)}{(\mathbf{k}_0\bullet\mathbf{d}_1)|\mathbf{k}_1|^2}\right\}\neq\phi, \tag{18}$$

which shows that the space GHI problem has solutions with two degrees of freedom.

(2) If $(\mathbf{r}',\mathbf{r}'',\mathbf{r}''')<0$ and h is sufficiently small, then $\mathbf{k}_0\bullet\mathbf{d}_1<0$, $\mathbf{k}_1\bullet\mathbf{d}_0<0$, so

$$c\in C:=\left\{c\in R:0>c>\frac{(\mathbf{b}_4-c_0\mathbf{d}_0,\mathbf{d}_1,\mathbf{k}_1)}{(\mathbf{k}_0\bullet\mathbf{d}_1)|\mathbf{k}_1|^2}\right\}\neq\phi. \tag{19}$$

It implies the similar conclusion. These imply the following result:

Theorem 3. Suppose that $\mathbf{r}=\mathbf{r}(s)\in C^5[0,l]$ is a curve with nonvanishing torsion at $s=0$, then there exists $H>0$ such that for $0<h<H$, the space GHI problem has solutions with two degrees of freedom.

Denote by $\tau(s)$ the torsion of the curve $\mathbf{r}=\mathbf{r}(s),s\in[0,l]$. If $\tau(s)\neq 0$ for $s\in[0,l]$, there exists $H(s)>0$ such that the space GHI problem has solution for the curve $\mathbf{r}(s)$ on $[s,s+H(s)]\cap[0,l]$ or $[s-H(s),s]\cap[0,l]$. Note that $[0,l]$ is a bounded and closed interval, from the Theorem of Finite Covering, we can select finite intervals from $\{s-H(s),s+H(s):s\in[0,l]\}$ to cover $[0,l]$. Thus the problem in section 2.1.1 can be dealt with by the following Corollary.

Corollary 1. Suppose that $\mathbf{r}=\mathbf{r}(s)\in C^5[0,l]$ is a curve with nonvanishing torsion anywhere, then there exists a piecewise cubic B-spline curve satisfying the GC^2 condition.

4 Approximation Order

Suppose that $\mathbf{r}=\mathbf{r}(s)\in C^5[0,l]$ is a curve with nonvanishing torsion at s=0 (s is the arc length parameter), and $\mathbf{b}=\mathbf{b}(t),t\in[0,1]$ is the GHI interpolant. Then these two curves can be represented by

$$\mathbf{b}(t)=[x(t),y(t),z(t)],\mathbf{r}(s)=[X(s),Y(s),Z(s)], \tag{20}$$

respectively. Since the curve $\mathbf{r}(s)$ has nonzero torsion at $s=0$, so $\mathbf{r}'(0)\neq\mathbf{0},\mathbf{b}'(0)\neq\mathbf{0}$. Without loss of generality, we assume that the first coordinate of $\mathbf{r}'(0)$ is nonzero, i.e., $X'(0)\neq 0$. Hence $X(s),s\in[0,h]$ is invertible if h is sufficiently small. Let

$$\mathbf{b}_1(t)=\mathbf{b}(t)=[x_1(t),y_1(t),z_1(t)],t\in[0,z],$$
$$\mathbf{b}_2(t)=\mathbf{b}(t)=[x_2(t),y_2(t),z_2(t)],t\in[z,1].$$

It is clear that $\mathbf{b}_1(t),\mathbf{b}_2(t)$ are cubic polynomials. Recall that[7]

$$\mathbf{b}'(t)=\sum_{i=0}^{3}\mathbf{b}_i^1N_{i,2}^1(t),t\in[0,1], \tag{21}$$

where

$$\mathbf{b}_0^1=\frac{3}{z}\Delta\mathbf{b}_0,\mathbf{b}_1^1=3\Delta\mathbf{b}_1,\mathbf{b}_2^1=3\Delta\mathbf{b}_2,\mathbf{b}_3^1=\frac{3}{1-z}\Delta\mathbf{b}_3,$$

and $N_{i,2}^1(t), i=0,1,2,3$ are defined on the knot vector $\mathbf{U}'=\{0,0,0,z,1,1,1\}$.

The conditions (18),(19) imply that c is in the neighborhood of the original point. Let $c\to 0$, we find

$$\begin{aligned}
&l_1 \to \frac{\sqrt{1-z}}{3}h+O(h^2),\\
&\Delta\mathbf{b}_0 = l_0\mathbf{r}'(0),\\
&\Delta\mathbf{b}_1 \to \frac{1}{3}\mathbf{r}'(0)h+O(h^2),\\
&\Delta\mathbf{b}_2 \to \frac{2-\sqrt{1-z}}{3}\mathbf{r}'(0)h+O(h^2),\\
&\Delta\mathbf{b}_3 \to \frac{\sqrt{1-z}}{3}\mathbf{r}'(0)h+O(h^2).
\end{aligned}$$

Therefore, if c,h are sufficiently small, the signatures of the first coordinate of $\Delta\mathbf{b}_i$ are the same as $X'(0)$, i.e., there exist functions x_1^{-1} and X^{-1} satisfying

$$x_1(x_1^{-1}(v))=v, X(X^{-1}(v))=v .$$

This provides

$$\mathbf{B}_1(v)=[v, y_1\circ x_1^{-1}(v), z_1\circ x_1^{-1}(v)], v\in[0,v_1], \tag{22}$$

$$\mathbf{R}(v)=[v, Y\circ X^{-1}(v), Z\circ X^{-1}(v)], v\in[0,v_1], \tag{23}$$

where $\circ$ denotes the composition of functions. By using the chain rule,

$$\begin{aligned}
&\mathbf{B}_1'(v)=\frac{1}{x_1'(t)}[x_1'(t), y_1'(t), z_1'(t)],\\
&\mathbf{B}_1''(v)=\frac{1}{[x_1'(t)]^3}[0, x_1'(t)y_1''(t)-x_1''(t)y_1'(t), x_1'(t)z_1''(t)-x_1''(t)z_1'(t)],
\end{aligned}$$

and the corresponding formulas hold for the derivatives of $\mathbf{R}(v)$. We get

$$\mathrm{B}_1(0)=\mathrm{R}(0), \mathrm{B}_1'(0)=\mathrm{R}'(0), \mathrm{B}_1''(0)=\mathrm{R}''(0). \tag{24}$$

Let

$$\begin{aligned}
&f_1(v)=y_1\circ x_1^{-1}(v)-Y\circ X^{-1}(v),\\
&f_2(v)=z_1\circ x_1^{-1}(v)-Z\circ X^{-1}(v).
\end{aligned}$$

The error of the first segment is bounded by

$$e_1(h)=\sqrt{2}\max|f_i(v)|, i=1,2. \tag{25}$$

From (24), we have

$$f_i(0)=f_i'(0)=f_i''(0)=0, i=1,2, \tag{26}$$

and $f_i'''(0)=O(h), i=1,2$ [4]. This implies the desired approximation order $e_1(h)=O(h^4)$ if the 4th derivatives of $f_1(v)$ and $f_2(v)$ are bounded, independently of h . By using the chain rule again, we have

$$\frac{d^4(y_1\circ x_1^{-1})}{dv^4}=\sum_i \left(y_1^{(i)}\prod\nolimits_v x_1^{j_v}\right)\Big/ (x_1')^{1+\sum j_v}, \tag{27}$$

$$\frac{d^4(z_1\circ x_1^{-1})}{dv^4}=\sum_i \left(z_1^{(i)}\prod\nolimits_v x_1^{j_v}\right)\Big/ (x_1')^{1+\sum j_v}. \tag{28}$$

According to (21) and the expansions in Section 3, we have

$$|\mathbf{b}_1'(t)|=O(h),$$

and the corresponding formulas hold for $|\mathbf{b}_1''(t)|$ and $|\mathbf{b}_1'''(t)|$. From these and (27), (28), the 4th derivatives of $f_1(v)$ and $f_2(v)$ are bounded. It can be also discussed for the case of the second segment in the same way using the interpolation conditions at the right endpoint.

Theorem 4. Suppose that $\mathbf{r}=\mathbf{r}(s)\in C^5[0,l]$ is a curve with nonvanishing torsion at s=0, then there exists $H>0$ such that for $0<h<H$, the space GHI problem has solutions with two degrees of freedom and the interpolant has the 4th order accuracy.

5 Examples

In this section we compare our method with the classical Hermite interpolation, and the two methods described in Refs.[3,5] respectively. First we consider the helix:

$$r(t)=(\cos(t),\sin(t),t), t\in[0,h], h=\frac{\pi}{2},\frac{\pi}{4},\frac{\pi}{8},\frac{\pi}{16},\frac{\pi}{32}.$$

We compute numerically the arc length, energy $E=\int|k(s)|^2 ds$ (s is the arc length parameter), global curvature $E=\int|k(s)|\mathrm{d}s$, average curvature $\bar{k}$, and the difference of curvature and torsion between the original curve and the interpolants by the Hermite interpolation, the scheme described in Ref.[3] and our interpolation scheme. The original curve and these interpolants are labeled by Helix, Hermite, Höllig, and CGHI respectively.

For $h=\pi/2$, these interpolants and the original curve are plotted in Fig.1. In fact the CGHI and Höllig approximate the Helix so well that they override it when displayed on the computer screen, while the Hermite is bad. Curvature and torsion errors for Helix are shown in Figs.2 and 3 respectively. The energy and other information of these interpolants are listed in Table 1. We can conclude that CGHI is better than Höllig since it can approximate the curvature and torsion. Moreover, Hermite is the worst.

The smaller h becomes, the better the Höllig and CGHI will be. However the Hermite becomes unstable and its curvature and torsion fluctuates very much. Worst of all, as $h=\pi/8,\pi/16,\pi/32$, the arc length and whole curvature of the Hermite interpolants become more and more bigger. In the following we only list the corresponding data in Table 2 for $h=\pi/16$. Compared with the Höllig, although the CGHI's approximation order is lower than Höllig's, the performance of CGHI is as good as that of the Höllig because it interpolates the curvature vectors at the endpoints, but the Höllig does not.

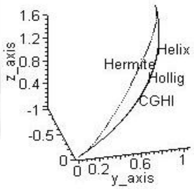

Fig.1 Interpolation curves for the helix with $\pi/2$

Table 1 $h=\pi/2, z=0.4, c=0.364$

Curves	s	E	K	$\bar{k}$
Helix	2.221 4	0.555 5	1.110 7	0.500 0
Hermite	2.176 8	0.730 6	1.092 0	0.501 6
Höllig	2.221 7	0.555 5	1.111 0	0.500 0
CGHI	2.221 5	0.555 5	1.110 7	0.500 0

Table 2 $h=\pi/2, z=0.5, c=1.3385$

Curves	s	E	K	$\bar{k}$
Helix	0.277 7	0.069 4	0.138 8	0.500 0
Hermite	0.682 7	20598.	6.422 2	9.406 7
Höllig	0.277 7	0.069 4	0.138 8	0.500 0
CGHI	0.277 7	0.069 2	0.138 8	0.500 0

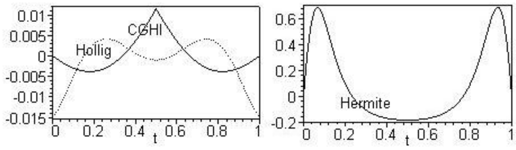

Fig.2 Curvature errors for the Helix with $\pi/2$

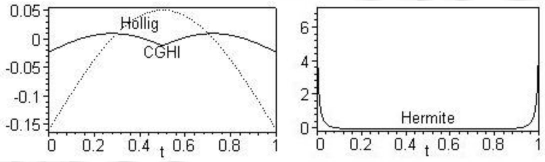

Fig.3 Torsion errors for the Helix with $\pi/2$

The ratios of consecutive errors defined by $m(h)=\log_2(e(h)/e(h/2))$ are listed in the table below for a sequence starting with $h=\pi/2$. Obviously, the convergence is of order $m=4$.

Table 3 The ratios of consecutive errors

h	e	m
$\pi/2$	0.3531×10^{-2}	
$\pi/4$	0.2201×10^{-3}	4.01
$\pi/8$	0.1375×10^{-4}	4.00
$\pi/16$	0.8589×10^{-6}	4.00
$\pi/32$	0.5374×10^{-7}	4.00

It should be pointed out that the B-spline has two free parameters z and c. They will be reduced to solve some nonlinear optimization problems. For $h=\pi/2$, the parameter c varies in C decided by (18). The energy and arc length of the B-spline dependent on c with $z=0.5$ are shown in Fig.4.

Our method can also be used to approximate the degree reduction of the splines. In order to approximate the Bézier of degree 5 with control points located on the cube (see Fig.5), we split it into two segments and compute the geometric Hermite interpolants using our method and two methods described in Refs.[3,5] for each segment. These interpolants are labeled by CGHI, Höllig, and Bézier respectively. The errors and the curvature errors are shown in Figs.6 and 7. We can see that the CGHI performs better than Höllig and Bézier while used to approximate the degree reduction in the example.

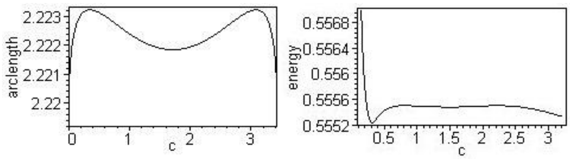

Fig.4 Energy and Arc length of B-spline depend on c

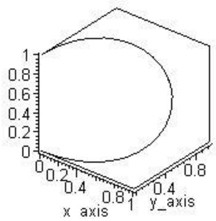

Fig.5 Bézier curve of degree 5

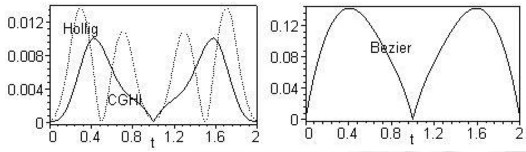

Fig.6 Errors of the interpolant curves for a Bézier curve of degree 5

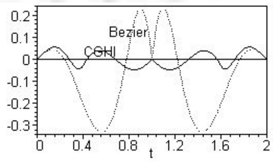

Fig.7 Curvature errors of interpolant curves for a Bézier curve of degree 5

6 Conclusions and Further Work

The geometric Hermite interpolation is the high accuracy approximation of smooth curves. Compared with the classical Hermite interpolation, the geometric Hermite interpolation has a great superiority since it is based on the geometric continuity and so it can drop down the degree of the interpolant without losing its geometric smoothness. Theoretical researches and examples show that the geometric Hermite interpolation has a very good approximation performance. Although our scheme is only $O(h^4)$ rather than $O(h^5)$ and $O(h^6)$ convergence rates of the schemes presented in Refs.[3,5], the performance of our scheme is very good compared with other two methods. Since it possesses two shape-parameters, we can control the shape-forming of the interpolant much easier. On the other hand, can we improve the approximation order from 4 to 5 or 6 by using two shape-parameters z,c? It is a difficult problem worthy of further studies.

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