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Shape Modification of Bézier Curves by Constrained Optimization*

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Abstract: Bézier curve is one kind of the most commonly used parametric curves in CAGD and Computer Graphics. Developing more convenient techniques for designing and modifying Bézier curve is an important problem. This paper investigates the optimal shape modification of Bézier curves by geometric constraints. A new method is presented in this paper by constrained optimization based on changing the control points of the curves. By this method, the authors modify control points of the original Bézier curves to satisfy the given constraints and modify the shape of the curves optimally. Practical examples are also given.

Key words: shape modification; Bézier curve; constrained optimization

Bézier curves are widely used in Computer Aided Geometric Design (CAGD) and Computer Graphics, and have many properties which are helpful for shape design. When Bézier curves are created, we often need to modify them to satisfy our design requirement.

Shape modification of NURBS or B-spline curves and surface has been attentively investigated. Piegl[1] proposed two methods to vary the shape of NURBS curves and surfaces: control-point-based modifiction and weight-based modification. Fowler and Bartels[2] gave a shape operator to force a curve or surface to assume the specified derivatives at selected parameter values. Au and Yuen[3] and Sánchez-Reyes[4] presented an approach for modifying the shape of NURBS curves by altering weights and the control points simultaneously. Hu *et al.*[5,6] developed a new method for shape modification of NURBS curves and surfaces with geometric constrained. However, developing more convenient way for shape modification of Bézier curves is still an important problem.

Inspired by Hu's results, we proposed a new method to modify the shape of Bézier curves by minimizing the changes of the shape in sense of least square. Explicit formulae are derived to compute positions of new control points of the modified curve with single point constraint. Shape modification of Bézier curve with added end point/tangent constraint, and multi-target points constraints are also discussed, we especially show the allowed area of target point in some special case.

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The paper is arranged as follows: problem statement is given in Section 1. Section 2 presents the constrained optimization method by minimizing changes of control points in Least-Square sense. Section 3 shows that shape modification with target points can be achieved by solving equation system. Practical examples and conclusion are given in Section 4.

1 Problem Statement

A Bézier curve of degree n can be defined as

$$\boldsymbol{P}(t)=\sum_{i=0}^{n}\boldsymbol{P}_i B_{i,n}(t),\quad 0\le t\le 1, \tag{1}$$

where P_i are control points, $B_{i,n}(t)$ are Bernstein function of degree n which can be defined as

$$B_{i,n}(t)=\frac{n!}{i!(n-i)!}t^i(1-t)^{n-i}=C_n^i t^i(1-t)^{n-i}. \tag{2}$$

As shown in Fig.1, T is a target point and S is a start point in curve $P(t)$ with parameter t_S . In order to let the curve pass through the target point T, we need to modify the curve.

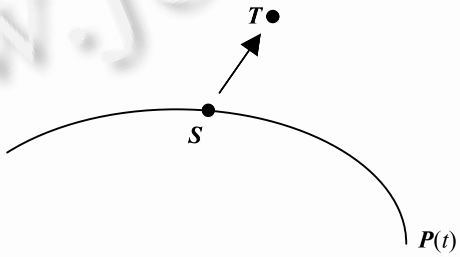

Fig.1 Illustration of local shape modification

2 Constrained Optimization Solution for Single Point Constraint

2.1 Single target point constraint

The distance between S and T is denoted by $D(S,T)$. A reasonable solution is to determine how many control points should be adjusted by the relationship between $D(S,T)$ and the shape of the control polygon. In general, we give explicit formulae for local shape modification by adjusting m control points (Here we suppose the end points P_0 and P_n are fixed for geometric continuity between adjacent curve segments), where m can be any number $1\le m\le(n-1)$.

Suppose the locations of control points $P_l,P_{l+1},\ldots,P_{l+m-1}$ are to be changed. We choose perturbation $\boldsymbol{\varepsilon}_i=(\varepsilon_i^x,\varepsilon_i^y,\varepsilon_i^z)^T$ for those m control points, so that the modified curve $\widetilde{\boldsymbol{P}}(t)$

$$\begin{aligned}\widetilde{\boldsymbol{P}}(t)&=\sum_{i=0}^{l-1}\boldsymbol{P}_i B_{i,n}(t)+\sum_{i=l}^{l+m-1}(\boldsymbol{P}_i+\boldsymbol{\varepsilon}_i)B_{i,n}(t)+\sum_{i=l+m}^{n}\boldsymbol{P}_i B_{i,n}(t)\\&=\sum_{i=0}^{n}\boldsymbol{P}_i B_{i,n}(t)+\sum_{i=l}^{l+m-1}\boldsymbol{\varepsilon}_i B_{i,n}(t)\end{aligned} \tag{3}$$

passes through the target point T, i.e., the curve satisfies the following equation

$$\boldsymbol{T}=\widetilde{\boldsymbol{P}}(t_s)=\sum_{i=0}^{n}\boldsymbol{P}_i B_{i,k}(t_s)+\sum_{i=l}^{l+m-1}\boldsymbol{\varepsilon}_i B_{i,k}(t_s)=\boldsymbol{S}+\sum_{i=l}^{l+m-1}\boldsymbol{\varepsilon}_i B_{i,k}(t_s). \tag{4}$$

We determine $\varepsilon_i\ (i=l,l+1,\ldots,l+m-1)$ by constrained optimization method, such that

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$$\sum_{i=l}^{l+m-1}\left\|\boldsymbol{\varepsilon}_i\right\|^2=\text{Min} \tag{5}$$

and the Lagrange function is defined by

$$L=\sum_{i=l}^{l+m-1}\left\|\boldsymbol{\varepsilon}_i\right\|^2+\boldsymbol{\lambda}\left(\boldsymbol{T}-\widetilde{\boldsymbol{P}}(t_s)\right) \tag{6}$$

where $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ is Langrange multiplier and $\|.\|$ is Euclidean norm. Let $\frac{\partial}{\partial\lambda_1}(L)=\frac{\partial}{\partial\lambda_2}(L)=\frac{\partial}{\partial\lambda_3}(L)=0$ and $\frac{\partial}{\partial\varepsilon_i^x}(L)=\frac{\partial}{\partial\varepsilon_i^y}(L)=\frac{\partial}{\partial\varepsilon_i^z}(L)=0$ for $i=l,l+1,...,l+m-1$, and write derived formula in vector form, the following equations can be derived

$$\begin{cases}\boldsymbol{T}=\boldsymbol{S}+\sum_{i=l}^{l+m-1}\varepsilon_i B_{i,n}(t_s),\\ \varepsilon_i=\frac{\lambda}{2}B_{i,n}(t_s), \qquad i=l,l+1,...,l+m-1.\end{cases} \tag{7}$$

By solving the above equation system, we can finally get the explicit solution as follows

$$\varepsilon_i=\frac{B_{i,n}(t_s)}{\sum_{j=l}^{l+m-1}B_{j,n}^2(t_s)}(\boldsymbol{T}-\boldsymbol{S}), \qquad i=l,l+1,...,l+m-1 \tag{8}$$

and the objective curve $\widetilde{\boldsymbol{P}}(t)$ can be obtained by substituting (8) into equation (3).

2.2 With added end point/tangent constraint

In many applications, we hope to keep the end point and its tangent for GC^1 continuity between adjacent curve segments. So control points P_0 and P_n should remain unchanged, and the new control points P'_1 and P'_{n-1} should be on the side P_1P_0, P_nP_{n-1} respectively. If control points P_1 and P_{n-1} remain unchanged, we can obtain explicit solution similarly. If P_1 and P_{n-1} are to be changed, there are $n-1$ control points which should be changed. We also choose perturbation $\boldsymbol{\varepsilon}_i=(\varepsilon_i^x,\varepsilon_i^y,\varepsilon_i^z)^T$ for those $n-1$ control points and suppose $\boldsymbol{\varepsilon}_1=\mu_1\frac{(\boldsymbol{P}_1-\boldsymbol{P}_0)}{\left\|\boldsymbol{P}_1-\boldsymbol{P}_0\right\|}$, $\boldsymbol{\varepsilon}_{n-1}=\mu_2\frac{(\boldsymbol{P}_n-\boldsymbol{P}_{n-1})}{\left\|\boldsymbol{P}_n-\boldsymbol{P}_{n-1}\right\|}$, the new modified curve can be defined as follows:

$$\widetilde{\boldsymbol{P}}(t)=\boldsymbol{P}_0B_{0,n}(t)+\left(\boldsymbol{P}_1+\mu_1\frac{(\boldsymbol{P}_1-\boldsymbol{P}_0)}{\left\|\boldsymbol{P}_1-\boldsymbol{P}_0\right\|}\right)B_{1,n}(t)+\sum_{i=2}^{n-2}(\boldsymbol{P}_i+\boldsymbol{\varepsilon}_i)B_{i,n}(t)+\left(\boldsymbol{P}_{n-1}+\mu_2\frac{(\boldsymbol{P}_n-\boldsymbol{P}_{n-1})}{\left\|\boldsymbol{P}_n-\boldsymbol{P}_{n-1}\right\|}\right)B_{n-1,n}(t)+\boldsymbol{P}_nB_{n,n}(t) \tag{9}$$

and the Langrange function can be defined as

$$L=\sum_{i=2}^{n-2}\left\|\boldsymbol{\varepsilon}_i\right\|^2+\left\|\mu_1\right\|^2+\left\|\mu_2\right\|^2+\boldsymbol{\lambda}\left(\boldsymbol{T}-\widetilde{\boldsymbol{P}}(t_s)\right). \tag{10}$$

So the following equations can be derived by constrained optimization method,

$$\begin{cases}\boldsymbol{T}-\boldsymbol{S}=\sum_{i=2}^{n-2}\boldsymbol{\varepsilon}_iB_{i,n}(t_s)+\mu_1\frac{(\boldsymbol{P}_1-\boldsymbol{P}_0)}{\left\|\boldsymbol{P}_1-\boldsymbol{P}_0\right\|}B_{1,n}(t_s)+\mu_2\frac{(\boldsymbol{P}_n-\boldsymbol{P}_{n-1})}{\left\|\boldsymbol{P}_n-\boldsymbol{P}_{n-1}\right\|}B_{n-1,n}(t_s),\\ \boldsymbol{\varepsilon}_i=\frac{1}{2}\boldsymbol{\lambda}B_{i,n}(t_s),\\ \mu_1=\frac{1}{2}\boldsymbol{\lambda}\frac{(\boldsymbol{P}_1-\boldsymbol{P}_0)}{\left\|\boldsymbol{P}_1-\boldsymbol{P}_0\right\|}B_{1,n}(t_s),\\ \mu_2=\frac{1}{2}\boldsymbol{\lambda}\frac{(\boldsymbol{P}_n-\boldsymbol{P}_{n-1})}{\left\|\boldsymbol{P}_n-\boldsymbol{P}_{n-1}\right\|}B_{n-1,n}(t_s).\end{cases} \tag{11}$$

Finally, the solution can be obtained by solving the equations above.

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2.3 A special case

If there are only four control points, the situation is very different. To satisfy the added tangent constrain at end point, the target point can not be located in anywhere. We hope to determine the allowed area. Firstly, all Bézier curves fall into three categories. The shapes of control nets are shown as Fig.2.

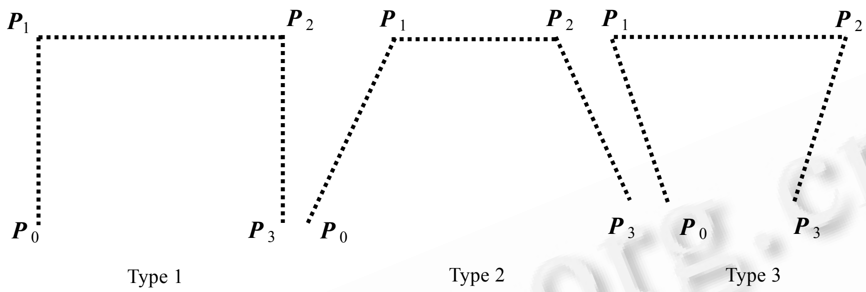

Fig.2 Three types of Bézier curve of degree 3

For a Bézier curve with control polygon of type 1, the target point can not be located outside the parallel line P_0P_1 and P_3P_2.

For a Bézier curve with control polygon of type 2, in order to keep the tangent of curve and end points, the target points can not be located at the left of the line P_0P_1 and the right of the line P_3P_2. Because the new control point P'_1 and P'_0 only can be set on the side P_0P_1 and P_3P_2, respectively, the target point can not be set beyond the curve determined by control points P_0, P_3, and the intersection point of side P_0P_1 and P_3P_2. Fig. 3(a) shows the allowed area of the target point.

For Bézier curve of type 3, it is similar with the type 2 yet only the two side P_0P_1 and P_3P_2 will not intersect at one point, so the target point can be located at the sector area determined by side P_0P_1 and P_3P_2 which is shown as Fig.3(b).

(a) (b)

Fig.3 Allowed area of target point for Bézier curve of type 2 and type 3

3 Multi-Target Point Constraints

For n order Bézier curve $P(t)$, if there is more than one target point, how to adjust control points so that the modified curve $P(t)$ passes through all those target points?

Suppose there are j+1 target points T_l, l=0,1,…,j. By projecting point T_l to curve $P(t)$, the corresponding parameter t_l can be obtained. Then we choose perturbation $\boldsymbol{\varepsilon}_i=(\varepsilon_i^x,\varepsilon_i^y,\varepsilon_i^z)^T$ for every control point P_i (also except P_0 and P_n), so that the modified curve

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$$\widetilde{\boldsymbol{P}}(t)=\boldsymbol{P}_0B_{0,n}(t)+\sum_{i=1}^{n-1}(\boldsymbol{P}_i+\boldsymbol{\varepsilon}_{\mathrm{i}})B_{i,n}(t)+\boldsymbol{P}_nB_{n,n}(t) \tag{12}$$

satisfies the requirement

$$\boldsymbol{T}_l=\widetilde{\boldsymbol{P}}(t_l)=\boldsymbol{P}_0B_{0,n}(t_l)+\sum_{i=1}^{n-1}(\boldsymbol{P}_i+\varepsilon_{\mathrm{i}})B_{i,n}(t_l)+\boldsymbol{P}_nB_{n,n}(t_l),\qquad l=0,1,...,j \tag{13}$$

We determine $\varepsilon_i\ (i=0,1,...,n)$ by constrained optimization method, such that

$$\sum_{i=0}^{n}\|\boldsymbol{\varepsilon}_i\|^2=\mathrm{Min} \tag{14}$$

and Lagrange function is defined as

$$L=\sum_{i=1}^{n-1}\|\boldsymbol{\varepsilon}_i\|^2+\sum_{l=0}^{j}\boldsymbol{\lambda}_l\left(\boldsymbol{T}_l-\widetilde{\boldsymbol{P}}(t_l)\right) \tag{15}$$

the following equation system can be derived

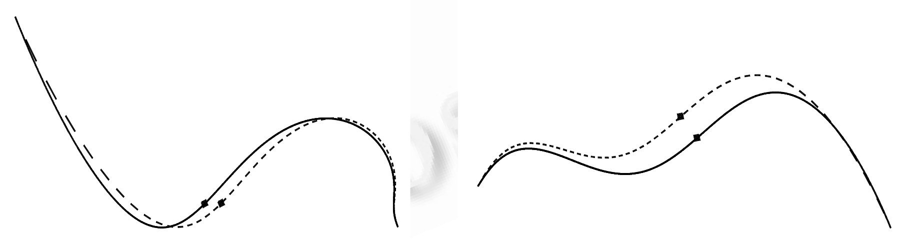
$$\begin{cases}\mathrm{T}_l=\sum_{i=0}^{n}\mathrm{P}_iB_{i,n}(t_l)+\sum_{i=1}^{n-1}\varepsilon_iB_{i,n}(t_l), & l=0,1,...,j\\ 2\varepsilon_i=\sum_{l=0}^{j}\lambda_lB_{i,n}(t_l), & i=1,2,...,n-1.\end{cases} \tag{16}$$

The constrained optimization solution can be obtained by solving the above equation system.

4 Practical Examples and Conclusion

We now give several examples to show the effects of the proposed method. In the following figures, original curves are shown as solid line, and modified curves are shown as dotted line. Figure 4 is for single point constraint, and Fig.5 is for multi-target point constraints.

The paper presents a method for shape modification of Bézier curve by minimizing changes of control points in Least-Square sense. Both single and multiple target points are considered. Practical examples show the proposed method is acceptable in CAD applications.



(a) (b)

Fig.4 Shape modification of Bézier curve with single target point

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(a) (b)

Fig.5 Shape modification of Bézier curve with multiple target points

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基于约束优化的 **Bézier** 曲线的形状修改

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摘要: 在计算机辅助几何设计和计算机图形学中,Bézier 曲线是一种常用的参数曲线,如何方便地设计和修改 Bézier 曲线是一个重要研究课题.研究了基于几何约束的 Bézier 曲线的优化的形状修改,提出一种基于修改曲线控制顶点的约束优化方法.该方法通过修改初始 Bézier 曲线的控制点来满足给定的约束,并理想地修改曲线的形状.同时给出了一些实例.

关键词: 形状修改;Bézier 曲线;约束优化

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