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有理Bézier三角曲面片低阶导矢界的估计*

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Bound Estimations on Lower Derivatives of Rational Triangular Bézier Surfaces

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Abstract: Based on the de Casteljau algorithm for triangular patches, also using some existing identities and elementary inequalities, this paper presents two kinds of new magnitude upper bounds on the lower derivatives of rational triangular Bézier surfaces. The first one, which is obtained by exploiting the diameter of the convex hull of the control net, is always stronger than the known one in case of the first derivative. For the second derivative, the first kind is an improvement on the existing one when the ratio of the maximum weight to the minimum weight is greater than 2; the second kind is characterized as being represented by the maximum distance of adjacent control points.
Key words: rational; triangular Bézier surface; de Casteljau algorithm; upper bound; intermediate weight; intermediate point

摘 要: 基于Bézier三角曲面的de Casteljau算法,同时运用一些恒等式和基本不等式,给出了两类有理Bézier三角曲面片低阶导矢的上界.第一类上界是用控制顶点凸包直径表示的,在一阶偏导的情况下,它是对已有上界的改进;在二阶偏导情况下,当最大权因子与最小权因子比值大于2时,它也是对已有上界的改进.第二类上界是用相邻控制顶点间距离的最大值来表示的.
关键词: 有理;Bézier三角曲面;de Casteljau算法;上界;中间权因子;中间点
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1 Introduction

Rational Bézier curves and surfaces are well established as a convenient way to represent Computer Aided

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Design Geometry. In some applications, it is important to have a measure of the size of the lower (first and second order) derivatives. For example, in order to detect the inflection points or singularities (cusps or loops) of curves or the flatness of surfaces, the magnitude scope analysis of the curves or surfaces' derivatives is inevitable. The efficiency of various algorithms for CAD models, e.g., the algorithms about collision detection or rendering, can be enhanced if the upper bounds on the curves and surfaces' lower derivatives can be calculated in advance. In practical applications, the stronger the upper bounds are, the more useful it will be.

So far, the calculation and the bound estimation of the derivatives of parametric curves and surfaces have been studied widely[1−6]. However, in case of rational parametric curves and surfaces, the results included in these papers only focus on the calculation and bound estimation of the first derivative. For rational surfaces, the calculation formulas and bound estimations are only derived on the tensor product patches, i.e., rectangular patches. The evaluation formulas and bound estimations for the rectangular patches, however, are ineffective for rational triangular patches, since the three parameters of triangular patches are not independent. Recently, by using the direction operator, Zhang[7] has obtained the lower derivatives and bound estimations of rational triangular Bézier surfaces.

Rational triangular Bézier surfaces are used wildly in CAGD and CAD nowadays because these surfaces take advantage over rectangular patches in many ways. For example, rational triangular Bézier surfaces are suitable for geometry modeling based on irregular and scattered data. By using surfaces constructed over non-degenerate triangular parameter domains, we can also avoid the degeneracy of rectangular patches[8−11]. Since rational triangular patches are playing an important role in CAGD and CAD, we are motivated to improve the magnitude upper bounds on their lower derivatives.

According to the de Casteljau algorithm for triangular patches, any point in the triangular Bézier surface can be obtained from repeated linear interpolation of control vertices. In this paper, we investigate the properties of the intermediate weights and intermediate points in the de Casteljau scheme by using some existing identities and elementary inequalities. Based on these properties, we obtain two kinds of upper bounds on triangular patches' lower derivatives. The first kind of bound estimations, which exploit the diameter of the convex hull of the control net, improve the corresponding result in Ref.[7] in the case of first derivative. For the second derivative, the first kind of bounds are stronger than the corresponding results in Ref.[7] when the ratio of the maximum weight to the minimum weight is greater than 2. The second type of bound estimations are characterized by using the local distance of the control net, namely, the maximum length of the edge of the triangles, to express the upper bounds. In applications, we can compute both kinds of bounds for each partial derivative, and choose the smaller one as the ultimate upper bound estimation. An application of surface rendering shows that the bounds obtained in this paper are useful in practical applications.

2 Preliminary

A rational triangular Bézier surface of degree n is defined as

$$\boldsymbol{R}(u,v,w)=\frac{\sum_{i+j+k=n}\omega_{i,j,k}\boldsymbol{R}_{i,j,k}B_{i,j,k}^{n}(u,v,w)}{\sum_{i+j+k=n}\omega_{i,j,k}B_{i,j,k}^{n}(u,v,w)}=\frac{P(u,v,w)}{W(u,v,w)},\ (u,v,w)\in T\ .$$

where $\boldsymbol{R}_{i,j,k}\in\Re^3$ are the control points, $\omega_{i,j,k}$ are positive weights, $B_{i,j,k}^{n}(u,v,w)=\frac{n!}{i!\,j!\,k!}u^i v^j w^k$ are Bernstein basis functions, and T:$\{(u,v,w)|u+v+w=1,\ 0\le u,v,w\le 1\}$ is the parametric domain. Since the three parameters are not independent, we represent the rational triangular Bézier surface equivalently as follows to ensure the partial derivatives make sense

$$\boldsymbol{R}(u,v)=\frac{\sum_{i+j=0}^{n}\omega_{i,j}\boldsymbol{R}_{i,j}B_{i,j}^{n}(u,v)}{\sum_{i+j=0}^{n}\omega_{i,j}B_{i,j}^{n}(u,v)}=\frac{P(u,v)}{W(u,v)},\ (u,v)\in D \tag{1}$$

where $\omega_{i,j}=\omega_{i,j,k}$, $\boldsymbol{R}_{i,j}=\boldsymbol{R}_{i,j,k}$, $B_{i,j}^{n}(u,v)=B_{i,j,k}^{n}(u,v,1-u-v)$, $D=\{(u,v)|u+v\le 1,\ u,v\ge 0\}$. For convenience, in the rest of this paper, we still write w instead of $1-u-v$. Based on the de Casteljau algorithm for the rational Bézier triangular patches[12], we can obtain some identities as follows

$$X_{i,j}^{r}=uX_{i+1,j}^{r-1}+vX_{i,j+1}^{r-1}+wX_{i,j}^{r-1} \tag{2}$$

$$X_{i,j}^{r}=u^{2}X_{i+2,j}^{r-2}+2uvX_{i+1,j+1}^{r-2}+2uwX_{i+1,j}^{r-2}+2vwX_{i,j+1}^{r-2}+v^{2}X_{i,j+2}^{r-2}+w^{2}X_{i,j}^{r-2} \tag{3}$$

where symbol $X_{i,j}^{r}$ represents $\omega_{i,j}^{r}$ or $\boldsymbol{P}_{i,j}^{r}\omega_{i,j}^{r}$, $\omega_{i,j}^{r}$ and $\boldsymbol{P}_{i,j}^{r}$ are intermediate weights and intermediate points in the r-th step of de Casteljau algorithm respectively. It follows that $\boldsymbol{P}_{i,j}^{0}=\boldsymbol{R}_{i,j}=\boldsymbol{R}_{i,j,k}$, $\omega_{i,j}^{0}=\omega_{i,j}=\omega_{i,j,k}$, $\boldsymbol{R}(u,v)=\boldsymbol{P}_{0,0}^{n}$, $W(u,v)=\omega_{0,0}^{n}$. In order to derive the properties of the intermediate weights and intermediate points, we first introduce some notations as follows:

$$W_{M}=\max_{i+j\le n}\omega_{i,j},\ W_{N}=\min_{i+j\le n}\omega_{i,j},\ W=\frac{W_{M}}{W_{N}},$$

$$V_{1}=\max_{i+j\le n-1}\left\{\frac{\max\{\omega_{i+1,j},\omega_{i,j+1},\omega_{i,j}\}}{\min\{\omega_{i+1,j},\omega_{i,j+1},\omega_{i,j}\}}\right\},$$

$$V_{2}=\max_{i+j\le n-2}\left\{\frac{\max\{\omega_{i+2,j},\omega_{i,j+2},\omega_{i,j},\omega_{i+1,j+1},\omega_{i+1,j},\omega_{i,j+1}\}}{\min\{\omega_{i+2,j},\omega_{i,j+2},\omega_{i,j},\omega_{i+1,j+1},\omega_{i+1,j},\omega_{i,j+1}\}}\right\},$$

$$P_{M}=\max_{\substack{i+j\le n\\ p+q\le n}}\|\boldsymbol{P}_{i,j}-\boldsymbol{P}_{p,q}\|,$$

$$L_{1}^{r}=\max_{i+j\le n-r-1}\{\|\boldsymbol{P}_{i+1,j}^{r}-\boldsymbol{P}_{i,j+1}^{r}\|,\|\boldsymbol{P}_{i+1,j}^{r}-\boldsymbol{P}_{i,j}^{r}\|,\|\boldsymbol{P}_{i,j+1}^{r}-\boldsymbol{P}_{i,j}^{r}\|\},$$

$$L_{2}^{r}=\max_{i+j\le n-r-2}\{\|\boldsymbol{P}_{i+2,j}^{r}-\boldsymbol{P}_{i,j+2}^{r}\|,\|\boldsymbol{P}_{i+2,j}^{r}-\boldsymbol{P}_{i,j}^{r}\|,\|\boldsymbol{P}_{i,j+2,k}^{r}-\boldsymbol{P}_{i,j}^{r}\|,$$
$$\|\boldsymbol{P}_{i+2,j}^{r}-\boldsymbol{P}_{i,j+1}^{r}\|,\|\boldsymbol{P}_{i,j+2}^{r}-\boldsymbol{P}_{i+1,j}^{r}\|,\|\boldsymbol{P}_{i,j}^{r}-\boldsymbol{P}_{i+1,j+1}^{r}\|,L_{1}^{r}\}.$$

Compute the ratios of the maximum weight to the minimum one for every triangle and every four adjacent triangles (see Fig.1) of the control net respectively, then V_1 and V_2 represent the maximum ratios respectively; L_1^r represents the maximum distance between arbitrary two adjacent intermediate points in the r-th step of the de Castetljau algorithm. Particularly, L_1^0 represents the maximum distance between arbitrary two adjacent control points; L_2^r represents the maximum distance between arbitrary two intermediate points in each four adjacent triangles in the r-th step of the de Casteljau algorithm (see Fig.1).

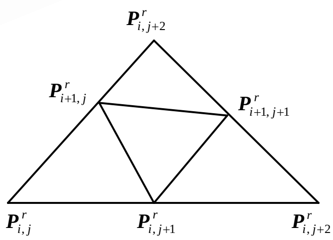

Fig.1 Four adjacent triangles in the r-th step of the de Casteljau algorithm

The identities derived in the rest of this section will be useful in the later discussion.

Applying $\dfrac{\partial B_{i,j}^n(u,v)}{\partial u}=n(B_{i-1,j}^{n-1}(u,v)-B_{i,j}^{n-1}(u,v))$ to $\boldsymbol{P}(u,v)$ and $W(u,v)$, we have

$$\begin{cases}\boldsymbol{P}_u=n(\omega_{1,0}^{n-1}\boldsymbol{P}_{1,0}^{n-1}-\omega_{0,0}^{n-1}\boldsymbol{P}_{0,0}^{n-1})\\ W_u=n(\omega_{1,0}^{n-1}-\omega_{0,0}^{n-1})\end{cases}\tag{4}$$

$$\begin{cases}\boldsymbol{P}_{uu}=n(n-1)(\omega_{2,0}^{n-2}\boldsymbol{P}_{2,0}^{n-2}-2\omega_{1,0}^{n-2}\boldsymbol{P}_{1,0}^{n-2}+\omega_{0,0}^{n-2}\boldsymbol{P}_{0,0}^{n-2})\\ W_{uu}=n(n-1)(\omega_{2,0}^{n-2}-2\omega_{1,0}^{n-2}+\omega_{0,0}^{n-2})\end{cases}\tag{5}$$

$$\begin{cases}\boldsymbol{P}_{uv}=n(n-1)(\omega_{1,1}^{n-2}\boldsymbol{P}_{1,1}^{n-2}-\omega_{1,0}^{n-2}\boldsymbol{P}_{1,0}^{n-2}-\omega_{0,1}^{n-2}\boldsymbol{P}_{0,1}^{n-2}+\omega_{0,0}^{n-2}\boldsymbol{P}_{0,0}^{n-2})\\ W_{uv}=n(n-1)(\omega_{1,1}^{n-2}-\omega_{1,0}^{n-2}-\omega_{0,1}^{n-2}+\omega_{0,0}^{n-2})\end{cases}\tag{6}$$

Substituting $\boldsymbol{R}(u,v)=\boldsymbol{P}_{00}^n$, $W(u,v,w)=\omega_{00}^n$, Eqs.(3), (4) and (5) into the first and second derivatives of Eq.(1) respectively, we obtain

$$\boldsymbol{R}_u=\frac{1}{W}(\boldsymbol{P}_u-\boldsymbol{R}W_u)=\frac{n}{\omega_{0,0}^n}[\omega_{1,0}^{n-1}(\boldsymbol{P}_{1,0}^{n-1}-\boldsymbol{P}_{0,0}^n)-\omega_{0,0}^{n-1}(\boldsymbol{P}_{0,0}^{n-1}-\boldsymbol{P}_{0,0}^n)]\tag{7}$$

$$\boldsymbol{R}_{uu}=\frac{(\boldsymbol{P}_{uu}-W_{uu}-2W_u\boldsymbol{R}_u)}{\omega_{0,0}^n}=\frac{n(n-1)}{\omega_{0,0}^n}(\omega_{2,0}^{n-2}(\boldsymbol{P}_{2,0}^{n-2}-\boldsymbol{P}_{0,0}^n)-2\omega_{1,0}^{n-2}(\boldsymbol{P}_{1,0}^{n-2}-\boldsymbol{P}_{0,0}^n)+\omega_{0,0}^{n-2}(\boldsymbol{P}_{0,0}^{n-2}-\boldsymbol{P}_{0,0}^n))-2\frac{W_u}{\omega_{0,0}^n}\boldsymbol{R}_u\tag{8}$$

$$\begin{aligned}\boldsymbol{R}_{uv}&=\frac{[(\boldsymbol{P}_{uv}-W_{uv}\boldsymbol{P}_{0,0}^n)-W_u\boldsymbol{R}_v-W_v\boldsymbol{R}_u)]}{\omega_{0,0}^n}\\&=\frac{n(n-1)}{\omega_{0,0}^n}(\omega_{1,1}^{n-2}(\boldsymbol{P}_{1,1}^{n-2}-\boldsymbol{P}_{0,0}^n)-\omega_{1,0}^{n-2}(\boldsymbol{P}_{1,0}^{n-2}-\boldsymbol{P}_{0,0}^n)-\omega_{0,1}^{n-2}(\boldsymbol{P}_{0,1}^{n-2}-\boldsymbol{P}_{0,0}^n)+\omega_{0,0}^{n-2}(\boldsymbol{P}_{0,0}^{n-2}-\boldsymbol{P}_{0,0}^n))-\frac{W_u}{\omega_{0,0}^n}(\boldsymbol{R}_u+\boldsymbol{R}_v)\end{aligned}\tag{9}$$

3 Properties of Intermediate Weights and Intermediate Points

As a preparation work for the bound estimations of rational triangular Bézier surfaces' lower derivatives in the next two sections, herein, we investigate the properties of intermediate weights ω_{ij}^r and intermediate points $\boldsymbol{P}_{ij}^r$. First, we introduce a lemma as follows.

Lemma 1. For arbitrary positive real numbers a,b and arbitrary vectors $\boldsymbol{p}_1,\boldsymbol{p}_2$, we have

$$|a\boldsymbol{P}_1-b\boldsymbol{P}_2|\le\max\{a,b\}\max\{|\boldsymbol{P}_1|,|\boldsymbol{P}_2|,|\boldsymbol{P}_1-\boldsymbol{P}_2|\}.$$

Proof: Without loss of generality, we suppose $a\le b$, and then we have

$$b-a\ge 0,$$

$$\begin{aligned}|a\boldsymbol{P}_1-b\boldsymbol{P}_2|&=|a(\boldsymbol{P}_1-\boldsymbol{P}_2)-(b-a)\boldsymbol{P}_2|\le(|a|+|b-a|)\max\{|\boldsymbol{P}_1-\boldsymbol{P}_2|,|\boldsymbol{P}_2|\}\\&\le b\max\{|\boldsymbol{P}_1-\boldsymbol{P}_2|,|\boldsymbol{P}_2|\}\le\max\{a,b\}\max\{|\boldsymbol{P}_1-\boldsymbol{P}_2|,|\boldsymbol{P}_1|,|\boldsymbol{P}_2|\}.\end{aligned}$$

Next, we discuss the properties of the intermediate weights.

Property 1. For intermediate weights of the r-th step and $(r-1)$-th step of the de Casteljau algorithm, we have

$$\frac{\omega_{i+1,j}^{r-1}}{\omega_{i,j}^r}\le A_1,\ \frac{\omega_{i,j+1}^{r-1}}{\omega_{i,j}^r}\le B_1,\ \frac{\omega_{i,j}^{r-1}}{\omega_{i,j}^r}\le C_1,$$

where $A_1=\max\limits_{i+j\le n-1}(\omega_{i+1,j}m_{ij}),\ B_1=\max\limits_{i+j\le n-1}(\omega_{i,j+1}m_{ij}),\ C_1=\max\limits_{i+j\le n-1}(\omega_{i,j}m_{ij}),\ m_{ij}=1\Big/\min\limits_{i+j\le n-1}\{\omega_{i,j+1},\omega_{i,j+1},\omega_{i,j}\}$.

Proof: For arbitrary positive real number a,b,c,d,e,f and $u,v\ge 0$, $u+v\le 1$, and note that $0\le w=1-u-v\le 1$, the following inequality always satisfies

$$\frac{ua+vb+wc}{ud+ve+wf}\le\max\left\{\frac{a}{d},\frac{b}{e},\frac{c}{f}\right\}.$$

Substituting $X_{i,j}^r$ by $\omega_{i,j}^r$ in Eq.(2) and applying this inequality repeatedly, we obtain

$$\frac{\omega_{i+1,j}^{r-1}}{\omega_{i,j}^{r}}=\frac{(u+v+w)\omega_{i+1,j,k}^{r-1}}{u\omega_{i+1,j,k}^{r-1}+v\omega_{i,j+1,k}^{r-1}+w\omega_{i,j,k+1}^{r-1}}\le\max\left\{1,\frac{\omega_{i+1,j}^{r-1}}{\omega_{i,j+1}^{r-1}},\frac{\omega_{i+1,j}^{r-1}}{\omega_{i,j}^{r-1}}\right\}\le A_1 .$$

The others can be obtained analogously.

Substituting $X_{i,j}^{r}$ by $\omega_{i,j}^{r}$ in Eq.(3), we have

$$\omega_{i,j}^{r}=u^2\omega_{i+2,j}^{r-2}+2uv\omega_{i+1,j+1}^{r-2}+2uw\omega_{i+1,j}^{r-2}+2vw\omega_{i,j+1}^{r-2}+v^2\omega_{i,j+2}^{r-2}+w^2\omega_{i,j}^{r-2} .$$ □

Based on this equation, we can get the following two properties in a similar fashion to that we prove Property 1.

Property 2. For intermediate weights of the *r*-th step and (*r*–2)-th step of the de Casteljau algorithm, we have

$$\frac{\omega_{i+2,j}^{r-2}}{\omega_{i,j}^{r}}\le A_2,\ \frac{\omega_{i,j+2}^{r-2}}{\omega_{i,j}^{r}}\le B_2,\ \frac{\omega_{i,j}^{r-2}}{\omega_{i,j}^{r}}\le C_2,\ \frac{\omega_{i,j+1}^{r-2}}{\omega_{i,j}^{r}}\le D_2,\ \frac{\omega_{i+1,j}^{r-2}}{\omega_{i,j}^{r}}\le E_2,\ \frac{\omega_{i+1,j+1}^{r-2}}{\omega_{i,j}^{r}}\le F_2 .$$

where

$$\begin{cases} A_2=\max\limits_{i+j\le n-2}(\omega_{i+2,j}M_{ij}),\ B_2=\max\limits_{i+j\le n-2}(\omega_{i,j+2}M_{ij}),\ C_2=\max\limits_{i+j\le n-2}(\omega_{i,j}M_{ij}),\\ D_2=\max\limits_{i+j\le n-2}(\omega_{i+1,j}M_{ij}),\ E_2=\max\limits_{i+j\le n-2}(\omega_{i+1,j}M_{ij}),\ F_2=\max\limits_{i+j\le n-2}(\omega_{i+1,j+1}M_{ij}),\\ M_{ij}=1\Big/\min\limits_{i+j\le n-2}\{\omega_{i+2,j},\omega_{i,j+2},\omega_{i,j},\omega_{i+1,j+1},\omega_{i+1,j},\omega_{i,j+1}\}. \end{cases}$$

Denote $V_1=\max\{A_1,B_1,C_1\}$, $V_2=\max\{A_2,B_2,C_2,D_2,E_2,F_2\}$, it is obvious that $1\le V_1\le V_2\le M$, therefore we have

Property 3. For intermediate weights of the *r*-th, (*r*–1)-th, (*r*–2)-th step of de Casteljau algorithm, we have

$$\frac{\omega_{i+1,j}^{r-1}}{\omega_{i,j}^{r}},\ \frac{\omega_{i,j+1}^{r-1}}{\omega_{i,j}^{r}},\ \frac{\omega_{i,j}^{r-1}}{\omega_{i,j}^{r}}\le V_1,$$
$$\frac{\omega_{i+2,j}^{r-2}}{\omega_{i,j}^{r}},\ \frac{\omega_{i,j+2}^{r-2}}{\omega_{i,j}^{r}},\ \frac{\omega_{i,j}^{r-2}}{\omega_{i,j}^{r}},\frac{\omega_{i,j+1}^{r-2}}{\omega_{i,j}^{r}},\ \frac{\omega_{i+1,j}^{r-2}}{\omega_{i,j}^{r}},\ \frac{\omega_{i+1,j+1}^{r-2}}{\omega_{i,j}^{r}}\le V_2.$$

In the rest of this section, we derive the properties of the intermediate points.

Property 4. For intermediate points of *r*-th and (*r*–1)-th step of de Casteljau algorithm, we have

$$L_1^r\le V_1L_1^{r-1} .$$

Proof: We prove $\|\boldsymbol{P}_{i+1,j}^{r}-\boldsymbol{P}_{i,j+1}^{r}\|\le V_1L_1^{r-1}$ as an example.

Recall Eq.(2), we have

$$\begin{aligned}(\boldsymbol{P}_{i+1,j}^{r}-\boldsymbol{P}_{i,j+1}^{r})=&\frac{1}{\omega_{i+1,j}^{r}}(u\boldsymbol{P}_{i+2,j}^{r-1}\omega_{i+2,j}^{r-1}+v\boldsymbol{P}_{i+1,j+1}^{r-1}\omega_{i+1,j+1}^{r-1}+w\boldsymbol{P}_{i+1,j}^{r-1}\omega_{i+1,j}^{r-1})+\\&\frac{1}{\omega_{i,j+1}^{r}}(u\boldsymbol{P}_{i+1,j+1}^{r-1}\omega_{i+1,j+1}^{r-1}+v\boldsymbol{P}_{i,j+2}^{r-1}\omega_{i,j+2}^{r-1}+w\boldsymbol{P}_{i,j+1}^{r-1}\omega_{i,j+1}^{r-1})\end{aligned} \tag{10}$$

Note that

$$v\omega_{i+1,j+1}^{r-1}=\omega_{i+1,j}^{r}-u\omega_{i+2,j}^{r-1}-w\omega_{i+1,j}^{r-1},$$
$$u\omega_{i+1,j+1}^{r-1}=\omega_{i,j+1}^{r}-v\omega_{i,j+2}^{r-1}-w\omega_{i,j+1}^{r-1}.$$

The substitution of these two identities into Eq.(10) implies

$$\begin{aligned}(\boldsymbol{P}_{i+1,j}^{r}-\boldsymbol{P}_{i,j+1}^{r})=&u\frac{\omega_{i+2,j}^{r-1}}{\omega_{i+1,j}^{r}}(\boldsymbol{P}_{i+2,j}^{r-1}-\boldsymbol{P}_{i+1,j+1}^{r-1})+v\frac{\omega_{i,j+2}^{r-1}}{\omega_{i+1,j}^{r}}(\boldsymbol{P}_{i+1,j+1}^{r-1}-\boldsymbol{P}_{i,j+2}^{r-1})+\\&w\left(\frac{\omega_{i+1,j}^{r-1}}{\omega_{i+1,j}^{r}}(\boldsymbol{P}_{i+1,j}^{r-1}-\boldsymbol{P}_{i+1,j+1}^{r-1})+\frac{\omega_{i,j+1}^{r-1}}{\omega_{i,j+1}^{r}}(\boldsymbol{P}_{i,j+1}^{r-1}-\boldsymbol{P}_{i+1,j+1}^{r-1})\right).\end{aligned}$$

Applying Lemma 1 and Property 3, we have

$$\|\boldsymbol{P}_{i+1,j}^{r}-\boldsymbol{P}_{i,j+1}^{r}\|\le\max\{A_1,B_1,C_1\}L_1^{r-1}\le V_1L_1^{r-1} .$$ □

Similarly, we can get

$$\| \boldsymbol{P}_{i+1,j}^{r} - \boldsymbol{P}_{i,j}^{r} \| \le V_1 L_1^{r-1},\ \ \| \boldsymbol{P}_{i,j+1}^{r} - \boldsymbol{P}_{i,j}^{r} \| \le V_1 L_1^{r-1}.$$

This property implies that the maximum distance between any two adjacent intermediate points of the r-th step of de Casteljau algorithm can be bounded by that of the $(r-1)$-th step multiplying a constant factor V_1.

Property 5. For the intermediate points of the r-th and $(r-2)$-th step of de Casteljau algorithm, we have

$$L_2^r \le V_2 L_2^{r-2}.$$

Proof: First, we prove $\| \boldsymbol{P}_{i+2,j}^{r} - \boldsymbol{P}_{i,j+2}^{r} \| \le V_2 L_2^{r-2}$. Following Eq.(3), we have

$$\begin{cases} \boldsymbol{P}_{i+2,j}^{r} = \dfrac{1}{\omega_{i+2,j}^{r}}(u^2 \boldsymbol{X}_{i+4,j}^{r-2} + v^2 \boldsymbol{X}_{i+2,j+2}^{r-2} + w^2 \boldsymbol{X}_{i+2,j}^{r-2} + 2uv \boldsymbol{X}_{i+3,j+1}^{r-2} + 2uw \boldsymbol{X}_{i+3,j}^{r-2} + 2vw \boldsymbol{X}_{i+2,j+1}^{r-2}) \\ \boldsymbol{P}_{i,j+2}^{r} = \dfrac{1}{\omega_{i,j+2}^{r}}(u^2 \boldsymbol{X}_{i+2,j+2}^{r-2} + v^2 \boldsymbol{X}_{i,j+4}^{r-2} + w^2 \boldsymbol{X}_{i,j+2}^{r-2} + 2uv \boldsymbol{X}_{i+1,j+3}^{r-2} + 2uw \boldsymbol{X}_{i+1,j+2}^{r-2} + 2vw \boldsymbol{X}_{i,j+3}^{r-2}) \end{cases} \tag{11}$$

where we write $\boldsymbol{X}_{i,j}^{r}$ instead of $\boldsymbol{P}_{i,j}^{r}\omega_{i,j}^{r}$ for brevity. Note that

$$v^2 \omega_{i+2,j+2}^{r-2} = \omega_{i+2,j}^{r} - u^2 \omega_{i+4,j}^{r-2} - w^2 \omega_{i+2,j}^{r-2} - 2uv \omega_{i+3,j+1}^{r-2} - 2uw \omega_{i+3,j}^{r-2} - 2vw \omega_{i+2,j+1}^{r-2},$$
$$u^2 \omega_{i+2,j+2}^{r-2} = \omega_{i,j+2}^{r} - v^2 \omega_{i,j+4}^{r-2} - w^2 \omega_{i,j+2}^{r-2} - 2uv \omega_{i+1,j+3}^{r-2} - 2uw \omega_{i+1,j+2}^{r-2} - 2vw \omega_{i,j+3}^{r-2}.$$

Substitute them into Eq.(11), we have

$$\begin{aligned} \boldsymbol{P}_{i+2,j}^{r} - \boldsymbol{P}_{i,j+2}^{r} = {} & u^2 \frac{\omega_{i+4,j}^{r-2}}{\omega_{i+2,j}^{r}}(\boldsymbol{P}_{i+4,j}^{r-2} - \boldsymbol{P}_{i+2,j+2}^{r-2}) + v^2 \frac{\omega_{i,j+4}^{r-2}}{\omega_{i,j+2}^{r}}(\boldsymbol{P}_{i,j+4}^{r-2} - \boldsymbol{P}_{i+2,j+2}^{r-2}) + \\ & w^2 \left(\frac{\omega_{i+2,j}^{r-2}}{\omega_{i+2,j}^{r}}(\boldsymbol{P}_{i+2,j}^{r-2} - \boldsymbol{P}_{i+2,j+2}^{r-2}) - \frac{\omega_{i,j+2}^{r-2}}{\omega_{i,j+2}^{r}}(\boldsymbol{P}_{i,j+2}^{r-2} - \boldsymbol{P}_{i+2,j+2}^{r-2}) \right) + \\ & 2uv \left(\frac{\omega_{i+3,j+1}^{r-2}}{\omega_{i+2,j}^{r}}(\boldsymbol{P}_{i+3,j+1}^{r-2} - \boldsymbol{P}_{i+2,j+2}^{r-2}) + \frac{\omega_{i+1,j+3}^{r-2}}{\omega_{i,j+2}^{r}}(\boldsymbol{P}_{i+1,j+3}^{r-2} - \boldsymbol{P}_{i+2,j+2}^{r-2}) \right) + \\ & 2uw \left(\frac{\omega_{i+3,j}^{r-2}}{\omega_{i+2,j}^{r}}(\boldsymbol{P}_{i+3,j}^{r-2} - \boldsymbol{P}_{i+2,j+2}^{r-2}) + \frac{\omega_{i+1,j+2}^{r-2}}{\omega_{i,j+2}^{r}}(\boldsymbol{P}_{i+1,j+2}^{r-2} - \boldsymbol{P}_{i+2,j+2}^{r-2}) \right) + \\ & 2vw \left(\frac{\omega_{i+2,j+1}^{r-2}}{\omega_{i+2,j}^{r}}(\boldsymbol{P}_{i+2,j+1}^{r-2} - \boldsymbol{P}_{i+2,j+2}^{r-2}) + \frac{\omega_{i,j+3}^{r-2}}{\omega_{i,j+2}^{r}}(\boldsymbol{P}_{i,j+3}^{r-2} - \boldsymbol{P}_{i+2,j+2}^{r-2}) \right). \end{aligned}$$

By applying Lemma 1 and Property 3, we have

$$\| \boldsymbol{P}_{i+2,j}^{r} - \boldsymbol{P}_{i,j+2}^{r} \| \le \max\{A_2, B_2, C_2, D_2, E_2, F_2\} L_2^{r-2} \le V_2 L_2^{r-2}.$$

Similarly, we have

$$\| \boldsymbol{P}_{i+2,j}^{r} - \boldsymbol{P}_{i,j}^{r} \|, \| \boldsymbol{P}_{i,j+2}^{r} - \boldsymbol{P}_{i,j}^{r} \|, \| \boldsymbol{P}_{i+2,j}^{r} - \boldsymbol{P}_{i,j+1}^{r} \|, \| \boldsymbol{P}_{i,j+2}^{r} - \boldsymbol{P}_{i+1,j}^{r} \|, \| \boldsymbol{P}_{i,j}^{r} - \boldsymbol{P}_{i+1,j+1}^{r} \| \le V_2 L_2^{r-2}. \qquad \square$$

The combination with Property 4 completes the proof.

This property implies that the maximum distance between arbitrary two points of all four adjacent triangles (see Fig.1) of the r-th step of de Casteljau algorithm can be bounded by that of the $(r-2)$-th step multiplying a constant factor V_2.

4 Bound Estimation by Using Control Net'S Convex Hull Diameter

In this section, we estimate the size of the derivatives by using the diameter of the control net's convex hull, which is denoted as P_M. We have the theorems as follows:

Theorem 1. For the first derivative of any rational triangular Bézier surface of degree n, we have

$$|\boldsymbol{R}_u| \le n\max\{A_1, C_1\} P_M \le n V_1 P_M.$$

Proof: From Eq.(7) and Lemma 1, it follows that

$$|\boldsymbol{R}_u|\le n\max\left\{\frac{\omega_{10}^{n-1}}{\omega_{00}^{n}},\frac{\omega_{00}^{n-1}}{\omega_{00}^{n}}\right\}\max\{|\boldsymbol{P}_{10}^{n-1}-\boldsymbol{P}_{00}^{n}|,|\boldsymbol{P}_{00}^{n-1}-\boldsymbol{P}_{00}^{n}|,|\boldsymbol{P}_{00}^{n-1}-\boldsymbol{P}_{10}^{n-1}|\},$$

where $\boldsymbol{P}_{00}^{n}$, $\boldsymbol{P}_{00}^{n-1}$, $\boldsymbol{P}_{10}^{n-1}$ are all included in the convex hull of the control net, therefore

$$|\boldsymbol{P}_{10}^{n-1}-\boldsymbol{P}_{00}^{n}|,|\boldsymbol{P}_{00}^{n-1}-\boldsymbol{P}_{00}^{n}|,|\boldsymbol{P}_{00}^{n-1}-\boldsymbol{P}_{10}^{n-1}|\le P_M.$$

Hence from Property 1, we have

$$|\boldsymbol{R}_u|\le n\max\{A_1,C_1\}P_M\le nV_1P_M \tag{12}$$

□

Similarly, from the last parts of Eq.(8), Eq.(9) and Lemma 1, we have:

Theorem 2. For the second derivative of any rational triangular Bézier surface of degree n in Eq.(1), we have

$$|\boldsymbol{R}_{uu}|\le[n(n-1)(A_2+2E_2+C_2)+2n^2(A_1+C_1)\max\{A_1,C_1\}]P_M,$$

$$|\boldsymbol{R}_{uv}|\le[n(n-1)(G_2+F_2+E_2+C_2)+n^2(A_1+C_1)\max\{A_1,C_1\}+n^2(B_1+C_1)\max\{B_1,C_1\}]P_M.$$

These results can also be simplified as

$$|\boldsymbol{R}_{uu}|,\ |\boldsymbol{R}_{uv}|\le 4n[(n-1)V_2+nV_1^2]P_M \tag{13}$$

5 Comparison

In this section, we compare the results proposed in previous section and the ones obtained in Ref.[7]. The upper bounds of the magnitudes of the first and second derivative proposed in Ref.[7] are showed as follows

$$|\boldsymbol{R}_u|\le nM^2P_M \tag{14}$$

$$|\boldsymbol{R}_{uu}|\le 2nM^2\left[(2n-1)M+n\frac{\max\limits_{i+j\le n-1}|\omega_{i,j}-\omega_{i+1,j}|}{W_n}\right]P_M \tag{15}$$

$$|\boldsymbol{R}_{uu}|\le 2nM^2\left[(2n-1)M+n\frac{\max\limits_{i+j\le n-1}|\omega_{i,j}-\omega_{i,j+1}|}{W_n}\right]P_M \tag{16}$$

It is obvious that bound (12) is an improvement on bound (14).

When $M\ge 2$, we have

$$\frac{4n[(n-1)V_2+nV_1^2]}{2nM^2\left[(2n-1)M+n\frac{\max\limits_{i+j+k=n-1}|\omega_{i,j,k+1}-\omega_{i+1,j,k}|}{W_n}\right]}\le\frac{2[(n-1)+nM]}{M^2(2n-1)}\le 1.$$

Therefore, we conclude that when $M\ge 2$, Eq.(13) gives a stronger bound than Eq.(15). Similarly, when $M\ge 2$, Eq.(13) also gives a stronger bound than Eq.(16). We denote the right parts of inequalities Eq.(13), Eq.(15), Eq.(16) as M_2, M_2, M_3 respectively. All together, by using the control net's convex hull diameter to estimate the size of the rational triangular Bézier surfaces' lower derivatives, we obtain the upper bounds as follows

$$|\boldsymbol{R}_u|\le n\max\{A_1,C_1\}P_M\le nV_1P_M,$$

$$|\boldsymbol{R}_{uu}|\le\begin{cases}M_1, & M\ge 2\\ \max\{M_1,M_2\} & M<2\end{cases},$$

$$|\boldsymbol{R}_{uv}|\le\begin{cases}M_1, & M\ge 2\\ \max\{M_1,M_3\} & M<2\end{cases}.$$

6 Bound Estimation by Using Local Distance

In this section, we establish another type of upper bounds, which only depend on the largest distance between the adjacent control points.

Theorem 3. For the first derivative of any rational triangular Bézier surface of degree *n*, we have

$$|\boldsymbol{R}_u| \le nV_1^{n+1}L_1^0 .$$

Proof: Let *r*=*n* in Eq.(2), we get two identities corresponding to X_{ij}^r representing ω_{ij}^r and $\boldsymbol{P}_{ij}^r\omega_{ij}^r$ respectively. Inserting them into Eq.(7), we have

$$\boldsymbol{R}_u = \frac{n(v\omega_{01}^{n-1}(\omega_{10}^{n-1}(\boldsymbol{P}_{10}^{n-1}-\boldsymbol{P}_{01}^{n-1})+\omega_{00}^{n-1}(\boldsymbol{P}_{01}^{n-1}-\boldsymbol{P}_{00}^{n-1}))+(1-v)\omega_{10}^{n-1}\omega_{00}^{n-1}(\boldsymbol{P}_{10}^{n-1}-\boldsymbol{P}_{00}^{n-1}))}{\omega_{00}^n\omega_{00}^n} .$$

Hence

$$|\boldsymbol{R}_u| \le nV_1^2L_M^{n-1} .$$

By repeatedly using Property 4, we complete the proof. □

Theorem 4. For the second derivative of every rational triangular Bézier surface of degree *n*, we have

$$|\boldsymbol{R}_{uu}|,|\boldsymbol{R}_{uv}| \le \begin{cases} 2n((n-1)V_2^{m+1}+2nV_1^{n+2})L_1^0 & n=2m \\ 2n((n-1)V_2^{m+1}+2nV_1^{n+1})V_1L_1^0 & n=2m+1 \end{cases} .$$

Proof: From Eq.(8), we have

$$\boldsymbol{R}_{uu} = \frac{1}{\omega_{00}^n\omega_{00}^n}(\boldsymbol{P}_{uu}\omega_{00}^n - W_{uu}\boldsymbol{P}_{00}^n\omega_{00}^n) - 2\frac{W_u\boldsymbol{R}_u}{\omega_{00}^n} \tag{17}$$

Expanding ω_{00}^n and $\boldsymbol{P}_{00}^n\omega_{00}^n$ according to Eq.(3) and substituting them into Eq.(17), then we have

$$\boldsymbol{R}_{uu} = n(n-1)(au^2+bv^2+cw^2+2duv+2euw+2fvw) - 2\frac{W_u\boldsymbol{R}_u}{\omega_{00}^n} ,$$

where *a*,*b*,*c*,*d*,*e*,*f* are linear combinations of $\omega_{pq}^{n-2}\omega_{st}^{n-2}(\boldsymbol{P}_{pq}^{n-2}-\boldsymbol{P}_{st}^{n-2})$ with $p+q\le 2, s+t\le 2$, for example,

$$a = \frac{1}{\omega_{00}^n\omega_{00}^n}(\omega_{20}^{n-2}(\omega_{20}^{n-2}(\boldsymbol{P}_{20}^{n-2}-\boldsymbol{P}_{20}^{n-2})-\omega_{10}^{n-2}(\boldsymbol{P}_{10}^{n-2}-\boldsymbol{P}_{20}^{n-2}))+\omega_{20}^{n-2}(\omega_{00}^{n-2}(\boldsymbol{P}_{00}^{n-2}-\boldsymbol{P}_{20}^{n-2})-\omega_{10}^{n-2}(\boldsymbol{P}_{10}^{n-2}-\boldsymbol{P}_{20}^{n-2}))) .$$

By using Lemma 1, Property 2 and Property 5, we have

$$|a| \le V_2^2L_2^{n-2} .$$

Similarly, we obtain

$$|b|,|c|,|d|,|e|,|f| \le V_2^2L_2^{n-2} .$$

Therefore, we have

$$|\boldsymbol{R}_{uu}| \le n(n-1)\max\{|a|,|b|,|c|,|d|,|e|,|f|\} + 4n^2V_1^{n+2}L_1^0 \le n(n-1)V_2^2L_2^{n-2} + 4n^2V_1^{n+2}L_1^0 .$$

Note that

$$L_2^1 \le 2L_1^1 \le 2V_1L_1^0,\ L_2^0 \le 2L_1^0 .$$

Hence, from Property 5, when *n*=2*m*, we have

$$|\boldsymbol{R}_{uu}| \le 2n[(n-1)V_2^{m+1}+2nV_1^{n+2}]L_1^0 .$$

Otherwise, when *n*=2*m*+1, we have

$$|\boldsymbol{R}_{uu}| \le 2n[(n-1)V_2^{m+1}+2nV_1^{n+1}]V_1L_1^0 .$$

The proof of the bound of $\boldsymbol{R}_{uv}$ can be similarly completed. □

7 An Application

In previous sections, we obtained two kinds of upper bounds, both of which could be computed. In applications, for each partial derivative, we can choose the lesser kind as the ultimate derivative bound estimation. As mentioned, derivative bounds of rational Bézier surfaces are useful in many areas. Here, we take the application in surface rendering as an example.

Triangular Bézier patches are commonly used to represent models for computer graphics, geometric modeling and animation. Large scale models, which are composed of tens of thousands of such surfaces, are commonly used to represent shapes of automobiles, building architectures, sculptured models and so on. They are also used in the applications involving surface fitting over scattered data or surface reconstruction. Many applications like interactive walkthroughs and design validation need to interactively visualize these surface models[13]. Hence, it is required to render these surfaces quickly and precisely. To this end, the choice of the global rendering step size becomes a crucial work, since unduly step size may result in excessive segments and defect further computations. However, step size is sometimes determined by the derivatives bounds. For example, suppose the step size in u and v directions are the same, we can compute the step size as follows by using Eq.(1) in Ref.[14]

$$s=\min\left\{\frac{2\sqrt{2\delta}}{\sqrt{|\boldsymbol{R}_{uu}^{i}|_{\max}+|\boldsymbol{R}_{uv}^{i}|_{\max}+|\boldsymbol{R}_{vv}^{i}|_{\max}}}\,\middle|\, i\in I\right\},$$

where δ is the tolerance, I is the set of the triangular patches and i is one patch with second derivative bounds being $|\boldsymbol{R}_{uu}^{i}|_{\max},|\boldsymbol{R}_{uv}^{i}|_{\max},|\boldsymbol{R}_{vv}^{i}|_{\max}$. It is obvious that the adoption of tighter bounds estimated in this paper can speed up the rendering process.

8 Conclusion

In this paper, we obtain two kinds of new upper bounds on triangular patches' lower derivatives by using the properties of the intermediate weights and intermediate points of the de Casteljau scheme as well as some existing identities and elementary inequalities. We get the first kind of bounds by using of the diameter of the convex hull of the control net. They are always stronger than the bounds obtained in Ref.[7] in the case of first derivative. In the case of second derivative, when the ratio of maximum weight to minimum weight is greater than 2, the first kind of bounds are more precise than the results in Ref.[7]; we compute the second kind of bounds by using the control net's local distance, i.e., the largest distance between arbitrary two adjacent points. In practical application, we can compute both kinds of upper bounds for each partial derivative, and choose the smaller one as the estimation of the derivative magnitudes. Example shows that the bounds estimated in this paper are useful in the practical applications.

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