

## Two Types of Polynomial Approximation to Rational Surfaces and Their Convergence\*

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**Abstract:** This paper investigates the relationship between the hybrid polynomial approximation and the Hermite polynomial approximation for rational surfaces. Under some assumptions of the control point weights, the paper derives the general necessary and sufficient conditions for which both the hybrid polynomial approximation and the Hermite polynomial approximation converge to the rational surface.

**Key words:** rational Bézier surface; polynomial approximation; Hermite approximation; hybrid approximation; convergence

Rational Bézier curves/surfaces are well established as a convenient way to represent Computer Aided Design geometry<sup>[1]</sup>. In mechanical engineering, contours that contain splines as well as conic sections cannot be stored in one data format if the curve description is restricted to polynomial curves. Moreover, it is difficult to communicate data between a CAD/CAM system based on rational Bézier curves/surfaces and another one that restricts itself to polynomial curves/surfaces.

Hybrid curves have been introduced in CAGD by Sederberg and Kakimoto<sup>[2]</sup>. They provide an attractive method for approximating rational Bézier curves by polynomial Bézier curves with the potential advantage that the size of the moving control point bounding boxes might be diminished as the degrees of the hybrid curves increase. Using this method Wang and Sederberg<sup>[3]</sup> computed the areas bounded by rational Bézier curves approximatively. Wang et al.<sup>[4]</sup> obtained the convergence condition for the hybrid polynomial approximation. Several methods were provided to estimate the error bounds for the approximation to the moving control point of the hybrid curve by Wang et al.<sup>[5]</sup>. Recently the authors have derived the formulae for calculating the control points of Hermite polynomial approximation to rational Bézier curves<sup>[6]</sup>. The above studies have enriched the theory of NURBS curves and given impetus to the application and development of NURBS curves in CAGD.

Qiu<sup>[7]</sup> generalized the case of approximating rational Bézier surfaces by the hybrid polynomial Bézier surfaces. But the generalization of the other work has not been found yet. In this paper we concentrate on the hybrid poly-

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mial approximation and the Hermite polynomial approximation to rational surface, their relationship and their convergence.

### 1 Polynomial Approximation to Rational Bézier Surfaces

A rational Bézier surface of degree  $n \times m$  is given by the following equation

$$R(u, v) = \frac{\sum_{k=0}^n \sum_{l=0}^m w_{kl} R_{kl} B_k^n(u) B_l^m(v)}{\sum_{k=0}^n \sum_{l=0}^m w_{kl} B_k^n(u) B_l^m(v)}, \tag{1}$$

where  $B_k^n(u) = C_k^n(1-u)^{n-k}u^k$  denotes the  $k$ -th Bernstein basis function of degree  $n$ ,  $B_l^m(v) = C_l^m(1-v)^{m-l}v^l$  denotes the  $l$ -th Bernstein basis function of degree  $m$ ,  $R_{kl}$  are the control points and  $w_{kl}$  are the control point weights. Throughout this paper the variable domain is  $0 \leq u \leq 1, 0 \leq v \leq 1$ .

The rational Bézier surface  $R(u, v)$  can be equivalently expressed as

$$R(u, v) = \tilde{H}^{r,p,s,q}(u, v) = \frac{\sum_{i=0}^{r+p} \sum_{j=0}^{s+q} \tilde{H}_{i,j}^{r,p,s,q}(u, v) B_i^{r+p}(u) B_j^{s+q}(v)}{\sum_{i=0}^{r+p} \sum_{j=0}^{s+q} \tilde{H}_{i,j}^{r,p,s,q}(u, v) B_i^{r+p}(u) B_j^{s+q}(v)} \tag{2}$$

where  $\tilde{H}_{i,j}^{r,p,s,q}(u, v)$  are rational Bézier surface of the same degrees and the same weights as the initial rational Bézier surface

$$\tilde{H}_{i,j}^{r,p,s,q}(u, v) = \frac{\sum_{k=0}^n \sum_{l=0}^m w_{kl} \tilde{H}_{i,j,k,l}^{r,p,s,q} B_k^n(u) B_l^m(v)}{\sum_{k=0}^n \sum_{l=0}^m w_{kl} B_k^n(u) B_l^m(v)}. \tag{3}$$

It is shown that

$$\tilde{H}_{i,j,k,l}^{r,p,s,q} = \begin{cases} \tilde{H}_{i,j}^{r,p,s,q} = H_{i,j}^{r,p,s,q}, & \text{if } i \neq r, j \neq s; \\ \tilde{H}_{r,i,k,l}^{r,p,s,q}, & \text{if } i = r, j \neq s, l = 0, 1, \dots, m; \\ \tilde{H}_{i,j,s,q}^{r,p,s,q}, & \text{if } i \neq r, j = s, k = 0, 1, \dots, n; \end{cases} \tag{4}$$

i. e.,  $\tilde{H}_{i,j,k,l}^{r,p,s,q} (i \neq r, j \neq s)$  are the fixed control points and  $\tilde{H}_{i,j}^{r,p,s,q} (i = r \text{ or } j = s)$  are the moving control points, the control points can be uniquely determined by (2)<sup>[7]</sup>. Such a polynomial Bézier surface with moving control points is called a hybrid surface. The formula (2) leads directly to a polynomial approximation to  $R(u, v)$  by simply replacing the moving points with stationary control points. If we replace  $\tilde{H}_{i,j,k,l}^{r,p,s,q} (i = r \text{ or } j = s)$  with constant control points  $H_{i,j}^{r,p,s,q} (i = r \text{ or } j = s)$  in the convex hull of  $\tilde{H}_{i,j,k,l}^{r,p,s,q} (i = r \text{ or } j = s)$ , then the Bézier surface  $H^{r,p,s,q}(u, v) = \sum_{i=0}^{r+p} \sum_{j=0}^{s+q} H_{i,j}^{r,p,s,q} B_i^{r+p}(u) B_j^{s+q}(v)$  gives a polynomial Bézier approximation to the rational Bézier surface  $R(u, v)$ . We call  $H^{r,p,s,q}(u, v)$  the degree  $(r+p) \times (s+q)$  hybrid polynomial approximation to  $R(u, v)$ .

Let

$$H^{r,p,s,q}(u, v) = \sum_{i=0}^{r+p-1} \sum_{j=0}^{s+q-1} H_{i,j}^{r,p,s,q} B_i^{r+p-1}(u) B_j^{s+q-1}(v) \tag{5}$$

The degree  $(r+p-1) \times (s+q-1)$  Bézier surface satisfies

$$\frac{\partial^{i+j} H^{r,p,s,q}(0,0)}{\partial u^i \partial v^j} = \frac{\partial^{i+j} R(0,0)}{\partial u^i \partial v^j}, \quad i=0,1,\dots,r-1, j=0,1,\dots,s-1; \tag{6}$$

$$\frac{\partial^{i+j} H^{r,p,s,q}(0,1)}{\partial u^i \partial v^j} = \frac{\partial^{i+j} R(0,1)}{\partial u^i \partial v^j}, \quad i=0,1,\dots,r-1, j=0,1,\dots,q-1; \tag{7}$$

$$\frac{\partial^{i+j} H^{r,p,s,q}(1,0)}{\partial u^i \partial v^j} = \frac{\partial^{i+j} R(1,0)}{\partial u^i \partial v^j}, \quad i=0,1,\dots,p-1, j=0,1,\dots,s-1; \tag{8}$$

$$\frac{\partial^{i+j} H^{r,p,s,q}(1,1)}{\partial u^i \partial v^j} = \frac{\partial^{i+j} R(1,1)}{\partial u^i \partial v^j}, \quad i=0,1,\dots,p-1, j=0,1,\dots,q-1. \tag{9}$$

There are  $(r+p) \times (s+q)$  linear equations about the  $(r+p) \times (s+q)$  variables  $H_{i,j}^{r,p,s,q} (i=0,1,\dots,r+p-1, j=0,$

1, ..., s+q-1) in Eqs. (6)~(9). We infer that  $h^{r,p+s,q}(u,v)$  is the unique polynomial surface with degree  $(r+p-1) \times (s+q-1)$  that satisfies the above Eqs. (6)~(9). It is called the Hermite polynomial approximation to  $R(u,v)^{[9]}$ . We say that  $h^{r,p+s,q}(u,v)$  is the degree  $(r+p-1) \times (s+q-1)$  Hermite polynomial approximation to  $R(u,v)$ .

**2 The Relationship between  $H^{r,p+s,q}(u,v)$  and  $h^{r,p+s,q}(u,v)$**

The hybrid approximation  $H^{r,p+s,q}(u,v)$  and the Hermite approximation  $h^{r,p+s,q}(u,v)$  are closely related.

**Theorem 1.** The control points of  $H^{r,p+s,q}(u,v)$  can be expressed by the control points of  $h^{r,p+s,q}(u,v)$  as

$$H_{i,j}^{r,p+s,q} = \left(1 - \frac{j}{s+q}\right) \left[ \left(1 - \frac{i}{r+p}\right) h_{i,j}^{r,p+s,q} + \frac{i}{r+p} h_{i-1,j}^{r,p+s,q} \right] + \frac{j}{s+q} \left[ \left(1 - \frac{i}{r+p}\right) h_{i,j}^{r,p+s,q} + \frac{i}{r+p} h_{i,j-1}^{r,p+s,q} \right], \quad i \neq r, j \neq s. \tag{10}$$

That is, if we raise the degree of polynomial  $h^{r,p+s,q}(u,v)$ , its coefficients and those of polynomial  $H^{r,p+s,q}(u,v)$  differ only in the case of  $i=r$  or  $j=s$ .

**Proof.** By simple calculation we get

$$\frac{\partial^{i+j} H^{r,p+s,q}(0,0)}{\partial u^i \partial v^j} = \frac{\partial^{i+j} h^{r,p+s,q}(0,0)}{\partial u^i \partial v^j}, \quad i=0,1,\dots,r-1, j=0,1,\dots,s-1. \tag{11}$$

Expanding both sides of the above equation, we arrive at

$$\frac{(r+p)!}{(r+p-i)!} \frac{(s+q)!}{(s+q-j)!} \sum_{k=0}^i \sum_{l=0}^j (-1)^{k+l} \binom{i}{k} \binom{j}{l} H_{-k,-l}^{r,p+s,q} = \frac{(r+p-1)!}{(r+p-1-i)!} \frac{(s+q-1)!}{(s+q-1-j)!} \sum_{k=0}^i \sum_{l=0}^j (-1)^{k+l} \binom{i}{k} \binom{j}{l} h_{-k,-l}^{r,p+s,q}. \tag{12}$$

The conclusion can be proved by mathematical induction on  $i$  and  $j$ . □

**Theorem 2.** The control points of  $h^{r,p+s,q}(u,v)$  can be expressed by the control points of  $H^{r-1,p-1,s-1,q-1}(u,v)$  as  $(0 \leq i \leq r-1, 0 \leq j \leq s-1)$

$$h_{i,j}^{r,p+s,q} = \left(1 - \frac{j}{s+q-1}\right) \left[ \left(1 - \frac{i}{r+p-1}\right) H_{i,j}^{r-1,p-1,s-1,q-1} + \frac{i}{r+p-1} H_{i-1,j}^{r-1,p-1,s-1,q-1} \right] + \frac{j}{s+q-1} \left[ \left(1 - \frac{i}{r+p-1}\right) H_{i,j}^{r-1,p-1,s-1,q-1} + \frac{i}{r+p-1} H_{i,j-1}^{r-1,p-1,s-1,q-1} \right], \tag{13}$$

$i=0,1,\dots,r-2, j=0,1,\dots,s-2;$

$$h_{r-1,j}^{r,p+s,q} = \left(1 - \frac{j}{s+q-1}\right) \left[ \frac{p}{r+p-1} \tilde{H}_{r-1,j,0,0}^{r-1,p-1,s-1,q-1} + \frac{r-1}{r+p-1} H_{r-2,j}^{r-1,p-1,s-1,q-1} \right] + \frac{j}{s+q-1} \left[ \frac{p}{r+p-1} \tilde{H}_{r-1,j,0,0}^{r-1,p-1,s-1,q-1} + \frac{r-1}{r+p-1} H_{r-2,j-1}^{r-1,p-1,s-1,q-1} \right], \quad j=0,1,\dots,s-2; \tag{14}$$

$$h_{i,s-1}^{r,p+s,q} = \frac{q}{s+q-1} \left[ \left(1 - \frac{i}{r+p-1}\right) \tilde{H}_{i,s-1,0,0}^{r-1,p-1,s-1,q-1} + \frac{i}{r+p-1} \tilde{H}_{i,s-1,0,0}^{r-1,p-1,s-1,q-1} \right] + \frac{s-1}{s+q-1} \left[ \left(1 - \frac{i}{r+p-1}\right) H_{i,s-1}^{r-1,p-1,s-1,q-1} + \frac{i}{r+p-1} H_{i,s-2}^{r-1,p-1,s-1,q-1} \right], \quad i=0,1,\dots,r-2; \tag{15}$$

$$h_{r-1,s-1}^{r,p+s,q} = \frac{q}{s+q-1} \left[ \frac{p}{r+p-1} \tilde{H}_{r-1,s-1,0,0}^{r-1,p-1,s-1,q-1} + \frac{r-1}{r+p-1} \tilde{H}_{r-1,s-1,0,0}^{r-1,p-1,s-1,q-1} \right] + \frac{s-1}{s+q-1} \left[ \frac{p}{r+p-1} H_{r-1,s-1}^{r-1,p-1,s-1,q-1} + \frac{r-1}{r+p-1} H_{r-2,s-2}^{r-1,p-1,s-1,q-1} \right]. \tag{16}$$

**Theorem 3.** The hybrid approximation is equivalent to the Hermite approximation under certain conditions as

$$H^{r,p+s,q}(u,v) = h^{r,p+s,q}(u,v) \quad \text{if} \quad \begin{cases} H_{i,j}^{r,p+s,q} = \tilde{H}_{i,j,0,0}^{r,p+s,q}, & i \neq r; \\ H_{r,j}^{r,p+s,q} = \tilde{H}_{r,j,0,0}^{r,p+s,q}, & j \neq s; \\ H_{r,s}^{r,p+s,q} = \tilde{H}_{r,s,0,0}^{r,p+s,q}. \end{cases} \tag{17}$$

The proofs of Theorem 2 and Theorem 3 are similar to the proof of Theorem 1, so we omit the details here.

### 3 Convergence Conditions for $H^{r,p^{i+q}}(u, v)$ and $h^{r,p^{i+q}}(u, v)$

#### 3.1 Bounds of remainder terms for $H^{r,p^{i+q}}(u, v)$ and $h^{r,p^{i+q}}(u, v)$

The following two theorems deal with the error terms of  $H^{r,p^{i+q}}(u, v)$  and  $h^{r,p^{i+q}}(u, v)$ .

**Theorem 4.** The remainder term for  $H^{r,p^{i+q}}(u, v)$  is bounded by

$$\|R(u, v) - H^{r,p^{i+q}}(u, v)\| \leq 4 \text{Max}_{i=r \text{ or } j=0,1,\dots,m} \text{Max}_{l=0,1,\dots,m} \|\tilde{H}_{i,j,k,l}^{r,p^{i+q}} - \tilde{H}_{i,j,0,0}^{r,p^{i+q}}\|. \tag{18}$$

Proof. Setting  $\Delta_{i,j}^{r,p^{i+q}} = \text{Max}_{k=0,1,\dots,k} \|\tilde{H}_{i,j,k,l}^{r,p^{i+q}} - H_{i,j}^{r,p^{i+q}}\|$  and  $\Delta_{\text{max}}^{r,p^{i+q}} = \text{Max}_{i=r \text{ or } j=0,1,\dots,m} \Delta_{i,j}^{r,p^{i+q}}$ , the remainder term for  $H^{r,p^{i+q}}(u, v)$  implies the inequality

$$\|R(u, v) - H^{r,p^{i+q}}(u, v)\| \leq (B_i^{i+q}(v) + B_j^{i+q}(u) - B_i^{i+q}(u)B_j^{i+q}(v)) \Delta_{\text{max}}^{r,p^{i+q}}. \tag{19}$$

Note that  $0 \leq B_i^{i+q}(u), B_j^{i+q}(v) \leq 1$  and

$$\|\tilde{H}_{i,j,k,l}^{r,p^{i+q}} - H_{i,j}^{r,p^{i+q}}\| \leq d(X) \{\tilde{H}_{i,j,k,l}^{r,p^{i+q}}\}_{k=0,1,\dots,m}^{l=0,1,\dots,m} \leq 2 \text{Max}_{i=0,1,\dots,m} \|\tilde{H}_{i,j,k,l}^{r,p^{i+q}} - H_{i,j}^{r,p^{i+q}}\| \tag{20}$$

where  $d(X)$  denotes the diameter of the convex hull of the set  $X$ . These together with the inequality (19) imply the result. □

**Theorem 5.** The remainder term for  $h^{r,p^{i+q}}(u, v)$  is bounded by

$$\|R(u, v) - h^{r,p^{i+q}}(u, v)\| \leq 4 \text{Max}_{l=0,1,\dots,m} \text{Max}_{j=0,1,\dots,m} \|\tilde{H}_{i,j,0,l}^{r,p^{i+q}} - H_{i,j,0,0}^{r,p^{i+q}}\|. \tag{21}$$

Proof. Using Theorem 4 and the fact that  $h^{r,p^{i+q}}(u, v) = H^{r-1,p^{i+q}}(u, v)$  under certain conditions, the inequality (21) is seen to be valid. □

#### 3.2 Recursive formula for $\tilde{H}_{i,j,k,l}^{r,p^{i+q}} - \tilde{H}_{i,j,0,0}^{r,p^{i+q}}$

For convenience of discussion we assume that  $w_{i,j} = g_{i,j}$ . By the expression of  $\tilde{H}_{i,j,k,l}^{r,p^{i+q}}$  as in Ref. [7] and making use of the fact that the control points  $\tilde{H}_{i,j,k,l}^{r,p^{i+q}}$  satisfy (4), we obtain the following result.

**Theorem 6.**  $\tilde{H}_{r+1,i+1,j}^{r,p^{i+q}} - \tilde{H}_{r+1,i,0,0}^{r,p^{i+q}}$  and  $\tilde{H}_{r,i,j,k,l}^{r,p^{i+q}} - \tilde{H}_{r,i,j,0,0}^{r,p^{i+q}}$  have the recursive formula

$$\begin{pmatrix} \tilde{H}_{r+1,i+1,j}^{r,p^{i+q}} - \tilde{H}_{r+1,i,0,0}^{r,p^{i+q}} \\ \tilde{H}_{r+1,i+1,2,l}^{r,p^{i+q}} - \tilde{H}_{r+1,i,0,0}^{r,p^{i+q}} \\ \vdots \\ \tilde{H}_{r+1,i+1,j}^{r,p^{i+q}} - \tilde{H}_{r+1,i,0,0}^{r,p^{i+q}} \end{pmatrix} = X_n^{r,p} \begin{pmatrix} \tilde{H}_{r,i,j,0,0}^{r,p^{i+q}} - \tilde{H}_{r,i,0,0,0}^{r,p^{i+q}} \\ \tilde{H}_{r,i,2,l}^{r,p^{i+q}} - \tilde{H}_{r,i,0,0,0}^{r,p^{i+q}} \\ \vdots \\ \tilde{H}_{r,i,j,0,0}^{r,p^{i+q}} - \tilde{H}_{r,i,0,0,0}^{r,p^{i+q}} \end{pmatrix}, l=0,1,\dots,m; \tag{22}$$

where  $X_n^{r,p} = \frac{2(r+1)(p+1)}{(r+p+1)(r+p+2)} \bar{X}^n$ ,  $\Gamma_k = \frac{\binom{n}{k+1} g_{k+1}}{\binom{n}{k} g_k}$ ,  $k=0,1,\dots,n-1$ , and

$$\bar{X}^n = \frac{1}{2} \begin{pmatrix} 2 - \Gamma_0 & \Gamma_1 & 0 & \dots & 0 & -\frac{1}{\Gamma_0} \\ \frac{1}{\Gamma_1} - \Gamma_0 & 2 & \Gamma_2 & \dots & 0 & -\frac{1}{\Gamma_1} \\ -\Gamma_0 & \frac{1}{\Gamma_2} & 2 & \dots & 0 & -\frac{1}{\Gamma_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\Gamma_0 & 0 & 0 & \dots & 2 & \Gamma_{r-1} - \frac{1}{\Gamma_{r-2}} \\ -\Gamma_0 & 0 & 0 & \dots & \frac{1}{\Gamma_{r-1}} & 2 - \frac{1}{\Gamma_{r-1}} \end{pmatrix}. \tag{23}$$

Similarly, using the expression of  $\tilde{H}_{i,j,k,l}^{r,p^{i+q+1}}$ , we get the following theorem.

**Theorem 7.**  $\tilde{H}_{r,s,k}^{r,p,q+1}$ ,  $\tilde{H}_{r,s,0}^{r,p,q+1}$  and  $\tilde{H}_{r,s,k}^{r,p,q}$ ,  $\tilde{H}_{r,s,0}^{r,p,q}$  have the recursive formula

$$\begin{pmatrix} \tilde{H}_{r,s+1,k,1}^{r,p,q+1} - \tilde{H}_{r,s+1,k,0}^{r,p,q+1} \\ \tilde{H}_{r,s+1,k,2}^{r,p,q+1} - \tilde{H}_{r,s+1,k,0}^{r,p,q+1} \\ \vdots \\ \tilde{H}_{r,s+1,k,m}^{r,p,q+1} - \tilde{H}_{r,s+1,k,0}^{r,p,q+1} \end{pmatrix}^T = \begin{pmatrix} \tilde{H}_{r,s,k,q}^{r,p,q} - \tilde{H}_{r,s,k,0}^{r,p,q} \\ \tilde{H}_{r,s,k,2}^{r,p,q} - \tilde{H}_{r,s,k,0}^{r,p,q} \\ \vdots \\ \tilde{H}_{r,s,k,m}^{r,p,q} - \tilde{H}_{r,s,k,0}^{r,p,q} \end{pmatrix}^T Y_m^q, \quad k=0,1,\dots,n, \quad (24)$$

where  $Y_m^q = \frac{2(s+1)(q+1)}{(s+q+1)(s+q+2)} \bar{Y}^m$ ,  $H_l = \begin{pmatrix} m \\ l+1 \end{pmatrix} h_{l+1}$ ,  $l=0,1,\dots,m-1$ , and

$$\bar{Y}^m = \frac{1}{2} \begin{pmatrix} 2-H_0 & \frac{1}{H_1} & H_0 & -H_0 & \dots & -H_0 & -H_0 \\ H_1 & 2 & \frac{1}{H_2} & \dots & 0 & 0 \\ 0 & H_2 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & \frac{1}{H_{r-1}} \\ -\frac{1}{H_0} & -\frac{1}{H_1} & -\frac{1}{H_2} & \dots & H_{r-1} - \frac{1}{H_{r-2}} & 2 - \frac{1}{H_{r-1}} \end{pmatrix}. \quad (25)$$

**3.3 Convergence conditions for  $H^{r,p,s,q}(u,v)$  and  $h^{r,p,s,q}(u,v)$**

In the contexts of Theorem 4 and Theorem 5,  $H^{r,p,s,q}(u,v)$  and  $h^{r,p,s,q}(u,v)$  must converge to  $R(u,v)$  uniformly on  $[0,1] \times [0,1]$  if we are given

$$\lim_{l \rightarrow \infty} \| \tilde{H}_{r,s,k,l}^{r,p,q} - \tilde{H}_{r,s,k,0}^{r,p,q} \| = 0, \quad k=0,1,\dots,n; \quad l=0,1,\dots,m; \quad r,j=0,1,\dots,2s. \quad (26)$$

Noting that  $B_s^{2s}(t)$  gets its maximum value at  $t = \frac{s}{2s} = \frac{1}{2}$ , with (19) we get

$$|R(u,v) - H^{r,p,s,q}(u,v)| \leq \left( 2 \binom{2s}{s} \left( \frac{1}{2} \right)^{2s} \right) \Delta_{\max}^{r,p,s,q}. \quad (27)$$

Taking the Stirling formula we obtain  $2 \binom{2s}{s} \left( \frac{1}{2} \right)^{2s} = O\left( \frac{1}{s} \right)$  as  $s \rightarrow \infty$ . The above analysis leads to the following observation.

**Theorem 8.** Let  $\Delta_{\max}^{r,p,s,q}$  be as defined in Theorem 4 and  $\sigma$  be a real constant such that  $0 < \sigma < \frac{1}{2}$ . Then:

$$\Delta_{\max}^{r,p,s,q} = O(s^\sigma), \quad s \rightarrow \infty, \quad (28)$$

is a sufficient condition for  $\tilde{H}^{r,p,s,q}(u,v)$  and  $h^{r,p,s,q}(u,v)$  to converge to  $R(u,v)$ .

By the assumption  $w_{r,s} = g_r h_s$ , we define  $g(u) = \sum_{k=0}^n g_k B_k^r(u)$  and  $h(v) = \sum_{l=0}^m h_l B_l^s(v)$ . With some mathematical techniques and skills, we can obtain a more interesting result. The following theorem gives the general necessary and sufficient conditions for  $H^{r,p,s,q}(u,v)$  and  $h^{r,p,s,q}(u,v)$  to converge to  $R(u,v)$ .

**Theorem 9.** The necessary and sufficient conditions under which  $H^{r,p,s,q}(u,v)$  and  $h^{r,p,s,q}(u,v)$  converge to  $R(u,v)$  as  $r+p \rightarrow \infty$  and  $s+q \rightarrow \infty$  are that all roots  $u_i (i=1,2,\dots,n)$  of  $g(u)$  and all roots  $v_j (j=1,2,\dots,m)$  of  $h(v)$  must satisfy

$$|u_i|^\alpha |1-u_i|^\beta > \alpha^\alpha \beta^\beta, \quad i=1,2,\dots,n, \quad (29)$$

and

$$|v_j|^\lambda |1-v_j|^\mu > \lambda^\lambda \mu^\mu, \quad j=1,2,\dots,m, \quad (30)$$

where  $\alpha = \lim_{r \rightarrow \infty} \frac{r}{r+p}$ ,  $\beta = 1-\alpha$ ,  $\lambda = \lim_{s \rightarrow \infty} \frac{s}{s+q}$ ,  $\mu = 1-\lambda$ .

#### 4 Conclusion

This paper is concerned with the hybrid polynomial approximation and the Hermite polynomial approximation to rational surface, their relationship and their convergence. Based on the notion of hybrid polynomials, we have derived the close relationship between the two types of polynomial approximation. The general necessary and sufficient convergence conditions for these polynomial approximations are also derived. We find the behavior of their convergence depends on the roots of the weight functions.

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## 有理曲面的两种多项式逼近及收敛性

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**摘要:** 研究了有理曲面的 hybrid 多项式逼近和 Hermite 多项式逼近的关系. 在权系数的某些假定下, 得到 hybrid 多项式逼近和 Hermite 多项式逼近均收敛的充分必要条件.

**关键词:** 有理 Bézier 曲面; 多项式逼近; Hermite 逼近; hybrid 逼近; 收敛性

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