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Relation of $\preceq$ in R/M and $\leqslant_T$ in R*

SUI Yue-fei

(Institute of Software The Chinese Academy of Sciences Beijing 100080)
E-mail: syf@ox.ios.ac.cn

Abstract It is proved that there are r.e. degrees **a** and **c** such that $[\mathbf{c}]\prec[\mathbf{a}]$ and $[\mathbf{b}]\neq[\mathbf{c}]$ for any r.e. degree $\mathbf{b}\leqslant_T\mathbf{a}$, where $[\mathbf{a}]$ is an element of **R/M**, the quotient of the recursively enumerable degrees **R** modulo the cappable degrees **M**.

Key words Recursively enumerable degree, weak truth table reduction.

1 Introduction

Ambos-Spies, Jockusch, Shore and Soare[1] proved that **M**, the set of all the cappable r.e. degrees, is an ideal in **R**; that **NC**, the set of all the noncappable r.e. degrees, is a filter in **R**; and that **NC**=**PS**, the set of all the promptly simple degrees. We have a quotient **R/M** of the r.e. degrees **R** modulo the cappable degrees **M**. An element in **R/M** is denoted by $[\mathbf{a}]$, the equivalence class of some r.e. degree **a** under the equivalence relation $\sim$, where $\mathbf{a}\sim\mathbf{b}$ iff

$$\exists\, \mathbf{c}_1,\ \mathbf{c}_2\in\mathbf{M}(\mathbf{a}\cup\mathbf{c}_1=\mathbf{b}\cup\mathbf{c}_2).$$

Given any $[\mathbf{a}],[\mathbf{b}]\in\mathbf{R/M}$, $[\mathbf{a}]\preceq[\mathbf{b}]$ if there is an r.e. degree $\mathbf{c}\in\mathbf{M}$ such that $\mathbf{a}\leqslant\mathbf{b}\cup\mathbf{c}$. $[\mathbf{a}]\prec[\mathbf{b}]$ if $[\mathbf{a}]\preceq[\mathbf{b}]$ and $[\mathbf{b}]\npreceq[\mathbf{a}]$. Let $[\mathbf{a}]\vee[\mathbf{b}]$ denote the least upper bound of $[\mathbf{a}]$ and $[\mathbf{b}]$. It is easy to prove that **R/M** is an upper semilattice, and $[\mathbf{a}]\vee[\mathbf{b}]=[\mathbf{a}\cup\mathbf{b}]$. Schwarz[2] proved the downward density theorem in **R/M**. Ambos-Spies (quoted in Ref. [3]) commented that the downward density theorem in **R/M** follows directly from the Robinson's splitting theorem and the fact that **NC**=**LC**, the set of all the r.e. degrees which cup to $\mathbf{0}'$ by low r.e. degrees.

By the definition of $\preceq$, given any r.e. degrees **a** and **c**, if $\mathbf{c}\leqslant\mathbf{a}$ then $[\mathbf{c}]\preceq[\mathbf{a}]$. Given any $[\mathbf{a}]$ and $[\mathbf{c}]\in\mathbf{R/M}$ such that $[\mathbf{c}]\preceq[\mathbf{a}]$, there is an r.e. degree $\mathbf{b}\in[\mathbf{a}]$ such that $\mathbf{c}\leqslant_T\mathbf{b}$. In this paper, we shall show that there are r.e. degrees **a** and **c** such that $[\mathbf{c}]\prec[\mathbf{a}]$ and for any r.e. degree $\mathbf{b}\leqslant_T\mathbf{a}$, $[\mathbf{b}]\neq[\mathbf{c}]$.

Our notation is standard, as described by Soare[4]. A number x is *unused at stage* $s+1$ if $x\geqslant s$ is greater than any number used so far in the construction. If the oracle is a join of two sets, we assume that the use is computed on the two sets separately, i.e., $\Gamma((A\oplus E)\lceil(\gamma(x)+1);x)=\Gamma(A\lceil(\gamma(x)+1)\oplus E\lceil(\gamma(x)+1);x)$, where $\gamma(x)$ is the use of $\Gamma(A\oplus E;x)$. If $\gamma(x)$ *moves to an unused number at stage* $s+1$, then $\gamma(x')$ moves for all $x'\geqslant x$, maintaining their order, to unused numbers. All use functions are assumed to be increasing in argument and nondecreasing in the stages.

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SUI Yue-fei was born in 1963. He received the Ph.D. degree in mathematics from the Institute of Software, the Chinese Academy of Sciences in 1988. He is a professor of the Institute of Software, the Chinese Academy of Sciences. His current research interests include recursion theory, theory of computability, complexity of computation, and rough set theory.

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2 Main Theorem, Its Requirements and the Priority Tree

Theorem 2.1. There exist r.e. degrees **a** and **c** such that [**c**]<[**a**], and for any r.e. degree $\mathbf{b}\leqslant_T\mathbf{a}$, $[\mathbf{b}]\neq[\mathbf{c}]$.

Proof. We shall construct r.e. sets A,C,B,E and define a recursive functional Γ such that B and E are a minimal pair, $C=\Gamma(A\oplus E)$, and the construction will satisfy for every $e\in\omega$ the following requirements:

$$\mathscr{P}_e: B\neq\omega-W_e,$$

$$\mathscr{M}_e: \Sigma_e(B)=\Sigma_e(E)=f_e \text{ total}\rightarrow f_e\leqslant_T\varnothing,$$

$$\mathscr{R}_e: D_e=\Phi_e(A)\&C=\Psi_e(D_e\oplus U_e)\&D_e=\Theta_e(C\oplus V_e)\rightarrow \deg_T(U_e)\in\mathbf{NC}\vee\deg_T(V_e)\in\mathbf{NC},$$

where $\{(D_e,U_e,V_e,\Phi_e,\Psi_e,\Theta_e)\}$ is a standard enumeration of all such sextuples (D,U,V,Φ,Ψ,Θ) that D,U,V are r.e. sets and Φ,Ψ,Θ are recursive functionals.

By **NC**=**PS** and the promptly simple degree theorem[1], we can decompose $\mathscr{R}_e$ into the following infinitely many subrequirements: for every $i,j\in\omega$,

$$\mathscr{R}_{e,i}: D_e=\Phi_e(A)\&C=\Psi_e(D_e\oplus U_e)\&D_e=\Theta_e(C\oplus V_e)$$
$$\&|W_i|=\infty\rightarrow\exists x\exists s(x\in W_{i,\mathrm{at}\,s}\&U_{e,s}\lceil x\neq U_{e,p_e(s)}\lceil x),$$

$$\mathscr{R}_{e,i,j}: \neg\mathscr{R}_{e,i}\&|W_j|=\infty\rightarrow\exists y\exists t(y\in W_{j,\mathrm{at}\,t}\&V_{e,t}\lceil y\neq V_{e,q_{e,i}(t)}\lceil y),$$

where $p_e, q_{e,i}$ are recursive functions defined in the construction to show the prompt simplicity of U_e and V_e, respectively.

The *priority tree* T is a subtree of $\Lambda^{<\omega}$, where $\Lambda=\{0,1,\mathbf{s},\mathbf{g},\mathbf{w}\}$. We define an order $<$ on T as follows:

$$\alpha<\beta\leftrightarrow\alpha\subset\beta\vee\exists\tau\subsetneq\alpha,\beta\exists a,b\in\Lambda(\tau\hat{\ }a\subseteq\alpha\&\tau\hat{\ }b\subseteq\beta\&a<_\Lambda b),$$

where $0<_\Lambda 1$, $\mathbf{s}<_\Lambda\mathbf{g}<_\Lambda\mathbf{w}$.

Given any node $\alpha\in T$, let S_α be the requirement assigned to α and T_α be the set of the remaining requirements that should be assigned to nodes $\supseteq\alpha$. Let S_α be the requirement in T_α with highest priority under a given linear order of all the requirements. We say that α is a *strategy* for S_α or an *S-strategy*. T_α is defined inductively as follows: Let $T_\lambda=\{\mathscr{P}_e,\mathscr{M}_e,\mathscr{R}_e: e\in\omega\}$.

Case 2.1. $S_\alpha=\mathscr{M}_e$. Then $\alpha\hat{\ }0$, $\alpha\hat{\ }1\in T$, and set $T_{\alpha\hat{\ }0}=T_\alpha-\{S_\alpha\}=T_{\alpha\hat{\ }1}$;

Case 2.2. $S_\alpha=\mathscr{R}_e$. Then $\alpha\hat{\ }0$, $\alpha\hat{\ }1\in T$, and set $T_{\alpha\hat{\ }0}=T_\alpha\cup\{\mathscr{R}_{e,i}: i\in\omega\}-\{S_\alpha\}$, $T_{\alpha\hat{\ }1}=T_\alpha-\{S_\alpha\}$;

Case 2.3. $S_\alpha=\mathscr{R}_{e,i}$. Then $\alpha\hat{\ }\mathbf{s}$, $\alpha\hat{\ }\mathbf{g}$, $\alpha\hat{\ }\mathbf{w}\in T$, and set $T_{\alpha\hat{\ }\mathbf{s}}=T_{\alpha\hat{\ }\mathbf{w}}=T_\alpha-\{S_\alpha\}$, $T_{\alpha\hat{\ }\mathbf{g}}=T_{\tau(\alpha)}\cup\{\mathscr{R}_{e,i,j}: j\in\omega\}-(\{\mathscr{R}_{e,i'}: i'\in\omega\}\cup\{S_{\tau(\alpha)},S_\xi: S_\xi \text{ is not an } \mathscr{R}\text{-strategy}\})$;

Case 2.4. $S_\alpha=\mathscr{P}_e$. Then $\alpha\hat{\ }\mathbf{w}$, $\alpha\hat{\ }\mathbf{s}\in T$, and set $T_{\alpha\hat{\ }\mathbf{w}}=T_{\alpha\hat{\ }\mathbf{s}}=T_\alpha-\{S_\alpha\}$;

Case 2.5. $S_\alpha=\mathscr{R}_{e,i,j}$. Then $\alpha\hat{\ }\mathbf{s}$, $\alpha\hat{\ }\mathbf{w}\in T$, and set $T_{\alpha\hat{\ }\mathbf{s}}=T_{\alpha\hat{\ }\mathbf{w}}=T_\alpha-\{S_\alpha\}$,

where

$$\tau(\alpha)=\begin{cases}\max\beta\hat{\ }\mathbf{g}\subseteq\alpha(S_\beta=\mathscr{R}_{e,i}) & \text{if } S_\alpha=\mathscr{R}_{e,i,j}\\ \max\beta\subseteq\alpha(S_\beta=\mathscr{R}_e) & \text{if } S_\alpha=\mathscr{R}_{e,i}.\end{cases}$$

Let δ_{s+1} be the last node we visit at stage $s+1$. At any stage $s+1$ we define the *length of agreement*:

$$l(\alpha,s)=\begin{cases}\max\{x:\forall y<x(\Sigma_{e,s}(E_s;y)=\Sigma_{e,s}(B_s;y)\downarrow)\} & \text{if } S_\alpha=\mathscr{M}_e\\ \max\{x:\forall y<x(D_{e,s}(y)=\Phi_{e,s}(A_s;y)\\ \quad\&C_s(y)=\Psi_{e,s}(D_{e,s}\oplus U_{e,s};y)\\ \quad\&D_{e,s}(y)=\Theta_{e,s}(C_s\oplus V_{e,s};y))\} & \text{if } S_\alpha=\mathscr{R}_e.\end{cases}$$

We assume that $l(\alpha,0)=0$ for every $\alpha\in T$.

$s+1$ is an *α-stage* if $\alpha\subseteq\delta_{s+1}$ and there is no open ξ-gap for any ξ with $\tau(\xi)\subset\alpha\subset\xi$; $s+1$ is *α-expansionary* if $s+1$ is an α-stage and $l(\alpha,s)>l(\alpha,t)$ for every α-stage $t+1\leqslant s$. α is *initialized* at stage $s+1$ if every parameter associated with α is set to be undefined.

3 Basic Modules

For every e,x,s, we shall use $\varphi_{e,s}(x),\psi_{e,s}(x)$ and $\theta_{e,s}(x)$ to denote $u(A_s;e,x,s),u(D_{e,s}\oplus U_{e,s};e,x,s)$ and $u(C_s\oplus V_{e,s};e,x,s)$, respectively.

To satisfy $\mathscr{P}_{e'}$, let η be a strategy for $\mathscr{P}_{e'}$. At any η-stage $s+1$, if there is a follower z of α such that $z\in W_{e',s}$, then we attempt to enumerate z in B.

The basic module for $\mathscr{M}_{e'}$ is the usual minimal pair strategy. Let ξ be a strategy for $\mathscr{M}_{e'}$. We shall preserve at least one side of these computations $\Sigma_{e,s}(B_s)\lceil l(\xi,s),\Sigma_{e,s}(E_s)\lceil l(\xi,s)$ for any ξ-expansionary stage $s+1$.

To satisfy $\mathscr{R}_{e,i,j}$, let α be a strategy for $\mathscr{R}_{e,i,j}$. Then there is a strategy β for $\mathscr{R}_{e,i}$ such that $\tau(\alpha)=\beta\hat{}\ g\subset\alpha$. We shall define a recursive function $p_{\tau(\beta)}$ to show the prompt simplicity of U_e and a recursive function $p_{\tau(\alpha)}$ to show the prompt simplicity of V_e.

At any $\tau(\beta)$-stage $s+1$, if there is an x such that $x\in W_{i,s}-W_{i,s'}$, and there is no β-gap at the last β-stage, where $s'+1\leqslant s$ is the last $\tau(\beta)$-stage, then open a β-gap, and if there is a y and a follower z_α of α such that

(3.1) $y\in W_{j,s}-W_{j,s''}$, where $s''+1\leqslant s$ is the last $\beta\hat{}\ g$-stage,

(3.2) there is no α-gap at the last α-stage, and

(3.3) $l(\tau(\beta),s)>z_\alpha$, $\psi_{e,s}(z_\alpha)<x$, $z_\alpha\notin C_s$, and $\theta_{e,s}(\psi_{e,s}(z_\alpha))<y$,

then open an α-gap. Enumerate $\gamma_s(z_\alpha)$ in A, and move $\gamma_{s+1}(z_\alpha)$ to an unused number.

Wait for the next $\tau(\beta)$-expansionary stage, say $t+1>s$. If $D_{e,s}\lceil\psi_{e,s}(z_\alpha)=D_{e,t}\lceil\psi_{e,s}(z_\alpha)$, then enumerate z_α in C, enumerate $\gamma_t(z_\alpha)$ in E and wait for the next $\tau(\beta)$-expansionary stage $u+1>t$. Then close the β-gap, $U_{e,s}\lceil\psi_{e,s}(z_\alpha)\neq U_{e,u}\lceil\psi_{e,s}(z_\alpha)$, define $p_{\tau(\beta)}(s)=u$, and $\mathscr{R}_{e,i}$ is *satisfied at* $\tau(\beta)$ unless $\tau(\beta)$ is initialized afterwards. In this case, α is initialized.

If $D_{e,s}\lceil\psi_{e,s}(z_\alpha)\neq D_{e,t}\lceil\psi_{e,s}(z_\alpha)$, then close the α-gap and the β-gap, define $p_{\tau(\alpha)}(s)=t,p_{\tau(\beta)}(s)=t$, and $V_{e,s}\lceil\theta_{e,s}(\psi_{e,s}(z_\alpha))\neq V_{e,t}\lceil\theta_{e,s}(\psi_{e,s}(z_\alpha))$, i.e., $V_{e,s}\lceil y\neq V_{e,t}\lceil y$; and $\mathscr{R}_{e,i,j}$ is *satisfied at* $\tau(\alpha)$ unless $\tau(\alpha)$ is initialized afterwards.

If there is a β-stage at stage $s+1$ and there are no such a y and a z_α, then at the next $\tau(\beta)$-expansionary stage, say $t+1>s$, close the β-gap, and define $p_{\tau(\beta)}(s)=t$.

There is a conflict between strategies. Let ξ be a strategy for $\mathscr{M}_{e'}$, let β be a strategy for $\mathscr{R}_{e,i}$ such that $\tau(\beta)\subset\xi\hat{}\ 0\subseteq\beta$. To satisfy a $\mathscr{P}$-strategy $\eta\supset\beta\hat{}\ g$, at any η-stage $s+1$, if η attempts to enumerate a follower z of η in B, then there is a β-gap and $s+1$ may not be a ξ-expansionary stage. To cope with it, at any η-stage $s+1$, if there is a follower z of η such that $z\in W_{e',s}$ and $s+1$ is ξ-expansionary then enumerate z in B; otherwise, define an auxiliary function $k:T\to T$ as follows: $k(\eta)=\xi$. At the next ξ-expansionary stage $t+1>s$, enumerate z in B.

Similarly, to solve the conflict between strategies for $\mathscr{R}_e$ and $\mathscr{M}_{e'}$, enumerate some $\gamma_s(z_\alpha)$ in E by a strategy for $\mathscr{R}_e$.

4 Construction

Stage $s=0$: Set $A_0=B_0=\varnothing$ and initialize every node $\alpha\in T$.

Stage $s+1$: Stage $s+1$ consists of at most s-many substages, $s_\lambda,\ldots,s_\alpha,\ldots$. At substage s_α we visit node α and do the following actions according to what strategy α is, and either δ_{s+1} is defined at s_α or the next node we shall visit at the next substage is defined, say $n(\alpha)$. If $|n(\alpha)|<s$, then go to substage $s_{n(\alpha)}$. If δ_{s+1} is defined or $|n(\alpha)|=s$, then s_α is the last substage of stage $s+1$. At the end of stage $s+1$, initialize every strategy $\gamma'>\delta_{s+1}$

with $\gamma' \supsetneq \delta_{s+1}$ and go to stage $s+2$.

Substage s_α: The procedure runs according to what strategy α is.

Case 4.1. Let α be a strategy for $\mathscr{P}_e$. If $\mathscr{P}_e$ is satisfied, i.e., there is a follower y of α, such that $y \in W_{e,s} \cap B_s$, then go to α ^ **s**; otherwise, if there is no follower of α, then assign the least unused number to be a follower of α, go to α ^ **w**; if there is a follower y of α such that $y \notin W_{e,s}$, then go to α ^ **w**; if there is a follower y of α such that $y \in W_{e,s}$, then let $\delta_{s+1}=\alpha$. Initialize every strategy $\gamma' > \alpha$ and see whether $s+1$ is β-expansionary for every $\mathscr{M}$-strategy β with β ^ $0 \subseteq \alpha$. If yes, then enumerate y in B; otherwise, define $k(\alpha)=\beta$ for the largest $\mathscr{M}$-strategy β such that β ^ $0 \subseteq \alpha$ and $s+1$ is not β-expansionary.

Case 4.2. Let α be a strategy for $\mathscr{R}_e$. If $s+1$ is α-expansionary and there is a strategy β for some $\mathscr{R}_{e,i,j}$ such that $\tau^2(\beta)=\alpha$ and β enumerated some element $\gamma_t(z_\beta)$ in A at the last β-stage $t+1$, where z_β is a follower of β, then let the β-gap be opened via some $y \in W_{j,t}$ and the $\tau(\beta)$-gap be opened via some $x \in W_{i,t}$. If $D_{e,s} \lceil \psi_{\beta,t}(z_\beta) = D_{e,t} \lceil \psi_{\beta,t}(z_\beta)$, then let $\delta_{s+1}=\beta$, and see whether $s+1$ is ξ-expansionary for every $\mathscr{M}$-strategy ξ with ξ ^ $0 \subseteq \alpha$. If yes, then enumerate z_β in C; enumerate $\gamma_s(z_\beta)$ in E, move $\gamma_{s+1}(z_\beta)$ to the least unused number; otherwise, define $k(\alpha)=\xi$ for the largest $\mathscr{M}$-strategy ξ such that ξ ^ $0 \subseteq \alpha$ and $s+1$ is not ξ-expansionary. If $D_{e,s} \lceil \psi_{\beta,t}(z_\beta) \neq D_{e,t} \lceil \psi_{\beta,t}(z_\beta)$, then close the $\tau(\beta)$-gap and the β-gap, define $p_\alpha(t')=s$ for every $t' \leqslant t$ with $p_{\alpha,s}(t') \uparrow$ and $p_{\tau(\beta)}(t'')=s$ for any $t'' \leqslant t$ with $p_{\tau(\beta),s}(t'') \uparrow$. If $V_{e,s} \lceil y \neq V_{e,t} \lceil y$, then $\mathscr{R}_{e,i,j}$ is satisfied at $\tau(\beta)$ unless $\tau(\beta)$ is initialized.

If $s+1$ is α-expansionary and there is a strategy β such that $\tau(\beta)=\alpha$, there is a β-gap via some x and there is no element enumerated in A by any strategy η with $\tau(\eta)=\beta$ at the last β-stage $t+1$, then close the β-gap, define $p_\alpha(t')=s$ for any $t' \leqslant t$ with $p_{\alpha,s}(t') \uparrow$. If $U_{e,t} \lceil x \neq U_{e,s} \lceil x$, then $\mathscr{R}_{e,i}$ is *satisfied at* α unless α is initialized.

If there is a strategy β for $\mathscr{R}_{e,i}$ such that $\tau(\beta)=\alpha$, $\mathscr{R}_{e,i}$ is not satisfied at α; there is an x such that $x \in W_{i,s} - W_{i,s'}$, where $s'+1 \leqslant s$ is the last α-stage; there is no strategy ξ with $\alpha \subset k(\xi) \subset \beta \subset \xi$; and there is no β-gap at the last β-stage, then let β be the least one and open a β-gap. If there is a strategy η for $\mathscr{R}_{e,i,j}$ such that $\tau(\eta)=\beta$ ^ **g**, $\mathscr{R}_{e,i,j}$ is not satisfied at $\tau(\eta)$ and there are a y and a follower z_η of η such that

(4.1) $y \in W_{j,s} - W_{j,s''}$, where $s''+1 \leqslant s$ is the last β ^ **g**-stage,

(4.2) there is no η-gap at the last η-stage,

(4.3) there is no strategy ξ such that $\beta \subset k(\xi) \subset \eta \subset \xi$,

(4.4) $l(\alpha,s) > z_\eta$, $z_\eta \notin C_s$, and

(4.5) $\psi_{e,s}(z_\eta) < x$, $\theta_{e,s}(\psi_{e,s}(z_\eta)) < y$,

then let η be the least one, open an η-gap; enumerate $\gamma_s(z_\eta)$ in A; move $\gamma_{s+1}(z_\eta)$ to the least unused number and initialize every strategy $\gamma' > \eta$. If there is no such η, then go to β ^ **g**.

If $s+1$ is α-expansionary and there is no such β, then go to α ^ 0; otherwise, go to α ^ 1.

Case 4.3. Let α be a strategy for $\mathscr{R}_{e,i}$. If $\mathscr{R}_{e,i}$ is satisfied at $\tau(\alpha)$, then go to α ^ **s**; otherwise, go to α ^ **w**.

Case 4.4. Let α be a strategy for $\mathscr{R}_{e,i,j}$. If $\mathscr{R}_{e,i,j}$ is satisfied at $\tau(\alpha)$, then go to α ^ **s**; otherwise, if there is no follower of α, then assign the least unused number to be a follower of α, go to α ^ **w**.

Case 4.5. Let α be a strategy for $\mathscr{M}_e$. If $s+1$ is α-expansionary and there is a strategy η such that $k(\eta)=\alpha$, then

(4.6) if $s+1$ is ξ-expansionary for every $\mathscr{M}$-strategy ξ with ξ ^ $0 \subseteq \alpha$, then if η is a $\mathscr{P}$-strategy and η attempted to enumerate some y in B at the last η-stage, then enumerate y in B; if η is an $\mathscr{R}$-strategy and η attempted to enumerate $\gamma_s(z_\beta)$ in E for some z_β at the last η-stage, then enumerate $\gamma_s(z_\beta)$ in E;

(4.7) otherwise, let $\delta_{s+1}=\alpha$, define $k(\eta)=\xi$, where ξ is the largest $\mathscr{M}$-strategy such that ξ ^ $0 \subseteq \alpha$ and $s+1$ is not ξ-expansionary.

If $s+1$ is α-expansionary and there is no such η, then go to α ^ 0; otherwise, go to α ^ 1.

This ends the description of the construction.

5 Verification

Let $\delta=\lim_s \inf\delta_s$ be the true path. We prove the following lemmas inductively on $\alpha\subset\delta$.

Lemma 5.1. If α is a strategy for $\mathscr{R}_{e,i,j}$ and β is a strategy for $\mathscr{R}_{e,i}$ such that $\tau(\alpha)=\beta^\frown \mathrm{g}\subset\alpha\subset\delta$, then $\mathscr{R}_{e,i,j}$ is satisfied at $\tau(\alpha)$ eventually, and α enumerates finitely many elements in A.

Proof. Let s_0 be the least α-stage after which α is not initialized. Let z_α be a follower of α at s_0.

Assume that $\mathscr{R}_{e,i,j}$ is not satisfied at $\tau(\alpha)$. Then W_i and W_j are infinite, and $\Phi_e(A)$, $\Psi_e(D_e\oplus U_e)$ and $\Theta_e(C\oplus V_e)$ are total. Since there are infinitely many $\tau(\beta)$-stages and $\tau(\alpha)$-gaps, we know that there are x,y and a stage $s+1>s_0$ such that (4.1)~(4.5) are satisfied and $\gamma_s(z_\alpha)$ is enumerated in A.

At the next $\tau(\beta)$-expansionary stage $t+1$, $D_{e,s}\lceil\psi_{e,s}(z_\alpha)\neq D_{e,t}\lceil\psi_{e,s}(z_\alpha)$; otherwise, z_α is enumerated in C eventually before the next $\tau(\beta)$-expansionary stage, say $u+1>t$. And by the initialization there is no element $<\psi_{e,s}(z_\alpha)$ enumerated in D_e, so $U_{e,s}\lceil\psi_{e,s}(z_\alpha)\neq U_{e,u}\lceil\psi_{e,s}(z_\alpha)$, i.e., $\mathscr{R}_{e,i}$ would be satisfied at $\tau(\beta)$ and $\beta^\frown \mathrm{s}\subset\delta$, a contradiction. By the initialization, there is no element $<\theta_{e,s}(\psi_{e,s}(z_\alpha))$ enumerated in C between s and t, so $V_{e,s}\lceil y\neq V_{e,t}\lceil y$, and $\mathscr{R}_{e,i,j}$ is satisfied at $\tau(\alpha)$, a contradiction.

Therefore, $\mathscr{R}_{e,i,j}$ is satisfied at $\tau(\alpha)$ eventually and α enumerates at most one element in A after s_0.

Lemma 5.2. If $\alpha\subset\delta$ is a strategy for $\mathscr{P}_e$, then $\mathscr{P}_e$ is satisfied eventually.

Proof. Let s_0 be the least α-stage after which α is not initialized. Let y be the follower of α at s_0.

Assume that P_e is not satisfied, i.e., $B=\omega-W_e$. If $y\notin W_e$, then $\mathscr{P}_e$ is satisfied, a contradiction. Assume that there exists an α-stage $s+1\geqslant s_0$ such that $y\in W_{e,s}$. If $s+1$ is ξ-expansionary for every $\mathscr{M}$-strategy ξ with $\xi^\frown 0\subseteq\alpha$, then y is enumerated in B at $s+1$ and $\mathscr{P}_e$ is satisfied, a contradiction; otherwise, for every $\mathscr{M}$-strategy ξ with $\xi^\frown 0\subseteq\alpha$, there is a ξ-expansionary stage $s_\xi+1$ such that no element is enumerated in E by any $\eta\supset\xi$ between $s_\xi+1$ and the stage when y is enumerated in B. Then, y is enumerated in B at $s_\xi+1$ for the $\mathscr{M}$-strategy ξ with least length and $\xi^\frown 0\subseteq\alpha$ and $\mathscr{P}_e$ is satisfied. A contradiction.

Therefore, $\mathscr{P}_e$ is satisfied eventually.

Lemma 5.3. If $\alpha\subset\delta$ is a strategy for $\mathscr{M}_e$, then $\mathscr{M}_e$ is satisfied.

Proof. Let s_0 be the least α-stage after which α is not initialized.

Assume that f_e is total. To recursively compute $f_e(x)$ for any given x, find the least α-expansionary stage $s+1\geqslant s_0$ such that $l(\alpha,s)>x$, and we claim that $f_e(x)=f_{e,s}(x)$.

We now claim that at any stage $t+1>s$, at least one of the two computations holds.

For a contradiction, suppose this fails at some least stage $t+1>s$, say, via a number z entering E or B. Then there is a stage $<t$ after the last α-expansionary stage when an element is enumerated in B or E, and one of the two computations is distroyed. Let $t_\alpha+1\leqslant t$ be the last α-expansionary stage.

If z enters E and destroys the remaining computation, then z enters E by an $\mathscr{R}$-strategy $\tau^2(\beta)$ and $z=\gamma_t(z_\beta)$, where z_β is a follower of β at stage $t+1$. If $\tau^2(\beta)\supseteq\alpha^\frown 0$, then by the definition of k, there is no element $<t_\alpha$ enumerated in B between $t_\alpha+1$ and $t+1$, a contradiction; if $\tau^2(\beta)>\alpha^\frown 0$ and $\tau^2(\beta)\not\supseteq\alpha^\frown 0$, then by the initialization, $z>t_\alpha+1\geqslant\sigma_{e,t_\alpha}(x)$, a contradiction. If $\tau^2(\beta)^\frown 0<\alpha$ and $\tau^2(\beta)^\frown 0\not\subseteq\alpha$, then α is initialized after s_0, a contradiction to the choice of s_0. The last possibility is that $\tau^2(\beta)^\frown 0\subseteq\alpha$. In this case, at the last β-stage, say $t''+1$, $\gamma_{t''}(z_\beta)$ was enumerated in A and $t+1$ is the next $\tau^2(\beta)$-expansionary stage, i.e., $t''\geqslant t_\alpha$. Hence, $\gamma_t(z_\beta)>t_\alpha$, since $\gamma_{t''}(z_\beta)$ moved to an unused number at $t''+1$. A contradiction.

If z enters B and destroys the remaining computation, then z enters B by a $\mathscr{P}$-strategy β. By the initialization and the choice of s_0, $\beta\supseteq\alpha^\frown 0$. By the definition of the auxiliary function k, z is enumerated in B at $t+1$ only if there is no element $<t_\alpha$ enumerated in E between $t_\alpha+1$ and $t+1$. Therefore, z cannot be enumerated in

B by β at $t+1$, a contradiction.

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R/M 中的$\preceq$与 R 中的$\leqslant_T$的关系

眭跃飞

(中国科学院软件研究所 北京 100080)

摘要 证明存在递归可枚举图灵度 a 和 c 使得 $c \preceq a$,并且对每个递归可枚举图灵度 $b \leqslant_T a$, $b \neq c$, 其中 a 是 R/M 中的一个元素,R/M 是递归可枚举图灵度集 R 模可盖图灵度集 M 的商.

关键词 递归可枚举度,弱真值表归约.

中图法分类号 TP301