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A High Diamond Theorem*

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Abstract It is shown that there exists a diamond of high computably enumerable degrees preserving the greatest element 1.

Key words Computability theory, computably enumerable set, computably enumerable degree, Turing reducibility, relative computability.

We say that a set $A \subseteq \omega$ is a *computably enumerable* (c.e.) set, if there is an algorithm to enumerate the elements of it. Turing[1] introduced the notion of *relative computability* between sets. For any sets $A, B \subseteq \omega$, we say that A is *computable in B* (or (Turing) *reducible to B*), if there is an algorithm to decide for any $x \in \omega$, whether $x \in A$ when answers are given to all questions of the form "Is $y \in B$?". We write $A \leqslant_T B$ to indicate A is computable in B, and $A \equiv_T B$ if $A \leqslant_T B$ and $B \leqslant_T A$. The equivalence class of A under $\equiv_T$ is the (Turing) *degree* of A and is written as $deg_T(A) = \mathbf{a}$. A degree $\mathbf{a}$ is called *computably enumerable* (c.e.) if it contains a c.e. set. The c.e sets and the corresponding degrees are central to computability theory. Post[2] noted that there is a greatest c.e. degree, written by $\mathbf{0}'$, and asked whether there is a c.e. degree other than $\mathbf{0}$ (the least degree) and $\mathbf{0}'$.

Friedberg[3] and independently Muchnik[4] answered the Post's problem affirmatively. Further, Sacks[5] showed that every nonzero c.e. degree can be (nontrivially) written as the least upper bound of two c.e. degrees, and Sacks[6] proved the density theorem of c.e. degrees. Shoenfield[7] then conjectured: for any finite partial orderings $P \subseteq Q$, with the least element 0 and the greatest element 1, any embedding of P into $\mathscr{E}$ (the set of all c.e. degrees) can be extended to an embedding of Q into the same $\mathscr{E}$. By this conjecture, there are no incomparable c.e. degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} \wedge \mathbf{b}$ (the greatest lower bound of $\mathbf{a}, \mathbf{b}$) exists. However, this was refuted by Lachlan[8] and independently Yates[9]: there exists a minimal pair of c.e. degrees, that is, there are incomparable c.e. degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$. Lachlan[8] also showed that both $\mathbf{a}$ and $\mathbf{b}$ can be chosen to be high. Cooper[10] showed that every high c.e. degree bounds a minimal pair, and Lachlan[11] showed the *Lachlan nonbounding theorem* that not every c.e. degree bounds a minimal pair. Recently, Li[12,13] proved the relative nonbounding theorem for halves of minimal pairs (i.e., cappables degrees).

A minimal pair yields an embedding of four-element Boolean algebra which is called *diamond* into $\mathscr{E}$

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preserving the least element **0**. Lachlan[8] showed that there are no c.e. degrees **a** and **b** such that $\mathbf{0}<\mathbf{a}<\mathbf{0}'$, $\mathbf{a}\vee\mathbf{b}=\mathbf{0}'$, and $\mathbf{a}\wedge\mathbf{b}=\mathbf{0}$. And Lachlan observed that for any incomparable c.e. degrees **a** and **b**, if $\mathbf{a}\vee\mathbf{b}=\mathbf{0}'$, then $\mathbf{a}\wedge\mathbf{b}$ (if it does exist) is not low. And Lachlan[8] and Robinson[14] suggested that perhaps c.e. degrees **a** and **b** with $\mathbf{a}\vee\mathbf{b}=\mathbf{0}'$ can never have an infimum. This was refuted by Shoenfield and Soare[15] and simultaneously by Lachlan[16]. Indeed, Lachlan[16] showed that every nonzero c.e. degree is the top of a diamond in c.e. degrees. And Slaman[17] showed that the diamond lattice is dense in c.e. degrees.

Ambos-Spies[18] showed that for any c.e. degrees $\mathbf{a}_i,\mathbf{b}_i(i=0,1)$, if $\mathbf{a}_0\vee\mathbf{a}_1=\mathbf{0}'$ and $\mathbf{b}_0\vee\mathbf{b}_1$ is low, then there are no c.e. degrees $\mathbf{c}_i\not\leqslant\mathbf{b}_i$ such that $\mathbf{a}_i\wedge\mathbf{c}_i\leqslant\mathbf{b}_i$, $i=0,1$. And again by this result for any c.e. degrees **a** and **b**, if $\mathbf{0}<\mathbf{a}<\mathbf{0}'$ and $\mathbf{a}\vee\mathbf{b}=\mathbf{0}'$ then $\mathbf{a}\wedge\mathbf{b}$ (if it exists) is not low. Fejer[19] showed that for any low c.e. degree **d**, there exist c.e. degrees **a**, $\mathbf{a}_0$, $\mathbf{a}_1$ such that $\mathbf{d}\leqslant\mathbf{a}<\mathbf{a}_0$, $\mathbf{a}_1<\mathbf{0}'$, $\mathbf{a}_0\vee\mathbf{a}_1=\mathbf{0}'$ and $\mathbf{a}_0\wedge\mathbf{a}_1=\mathbf{a}$. By analysing the *Lachlan nonsplitting theorem* (Lachlan[20]), Harrington[21] noted that there exists an incomplete c.e. degree **a** such that for any c.e. degrees **x**,**y**, if $\mathbf{x},\mathbf{y}\in[\mathbf{a},\mathbf{0}')$, then $\mathbf{0}'\neq\mathbf{x}\vee\mathbf{y}$ while the **a** is called *a Harrington nonsplitting base*. Further, Cooper and Li[22] showed that there exists a Low_3 Harrington nonsplitting base. On the other hand, Yi showed that there is a c.e. degree $\mathbf{a}\neq\mathbf{0}'$ such that for any c.e. degree $\mathbf{x}\in[\mathbf{a},\mathbf{0}')$, $\mathbf{0}'$ splits over **x**.

Y. Jiang and Z. Jiang asked (private communication): Is there a diamond of high c.e. degrees preserving the greatest element 1? In this paper, we give the answer.

Theorem * (High Diamond Theorem**). There exists a diamond of high c.e. degrees preserving 1, that is, there exist c.e. degrees **a**, $\mathbf{a}_0$, $\mathbf{a}_1$ such that **a** is high, $\mathbf{a}<\mathbf{a}_0<\mathbf{0}'$, $\mathbf{a}_0\vee\mathbf{a}_1=\mathbf{0}'$ and $\mathbf{a}_0\wedge\mathbf{a}_1=\mathbf{a}$.

In relation to this, Cooper and Li[23] have shown that there is no low_2 c.e. degree which is the bottom of a diamond of c.e. degrees with top $\mathbf{0}'$, and that there is a low_3 c.e. degree which is the bottom of a diamond of c.e. degrees with top $\mathbf{0}'$.

We organise the paper as follows. In section 1, we formulate the theorem by requirements and describe the basic modules for satisfying the requirements. In section 2, we describe the strategies based on the basic modules. In section 3, we arrange the strategies on nodes of a priority tree. In section 4, we describe the full construction. And in section 5, we verify that the construction satisfies all of the requirements.

Our notation and terminology are standard and generally follow Soare[24]. During the course of a construction, notations such as A_i, Φ_e are used to denote the current approximation to these objects. The notations $A_{i,s}$, $\Phi_e[s]$ denote the approximation to these objects which exist at the end of stage s. And the notation such as $\varphi_e[t,s]$ denotes the approximation of φ_e which exists at the end of substage t of stage s. The notation $[t,s]$ denotes a *state* of substage t of stage s, we define $[t,s]<[t',s']$ if either $s<s'$ or $s=s'$ and $t<t'$. The use function for a partial computable (p.c.) functional is the greatest number in its "oracle" which is actually used in the computation. For a p.c. functional, say Λ, which is not built by us, then for any x and s, if $\Phi(x)[s]\uparrow$, then define $\varphi(x)[s]=-1$. And for a p.c. functional, say Φ, which is built by us, then for any x and s, if $\Lambda(x)[s]\uparrow$, define $\lambda(x)[s]=\omega$. During the course of a construction, we say that y is *fresh* if y is the least number which is greater than any number mentioned so far.

1 The Requirements and the Basic Modules

1.1 The requirements

To prove the theorem, we construct c.e. sets A, A_0 and A_1, to satisfy the following requirements:

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* This result is also deduced from the above result of Yi and a claim of Downey and Shore that for any incomparable c.e. degrees **a**,**b**, there are c.e. degrees **c**,**d** such that $\mathbf{a}<\mathbf{c}<\mathbf{a}\vee\mathbf{b}$, $\mathbf{b}<\mathbf{d}<\mathbf{a}\vee\mathbf{b}$, and $\mathbf{c}\wedge\mathbf{d}$ exists.

** P. Zheng and A. Li have noted that $\mathbf{0}'$ here can be replaced by an arbitraily given high c.e. degree.

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$\mathcal{R}$: $K \leqslant_T A_0 \oplus A_1$;

$\mathcal{P}_e^i$: $A_{1-i} \neq \Phi_e(A)$;

$\mathcal{M}_e$: $\Psi_e^0(A_0, A) = \Psi_e^1(A_1, A) = g_e$ total $\to g_e \leqslant_T A$;

$\mathcal{I}_e$: $A^{[2e]} =^* J^{[2e]}$,

where $e \in \omega$, $i = 0, 1$, J is a fixed c.e. set such that for any $e \in \omega$, both (a) and (b) below hold:

(a) $e \in K' \to J^{[2e]}$ is finite;

(b) $e \notin K' \to J^{[2e]} = \omega^{[2e]}$.

K is a fixed creative set, and $\{(\Phi_e, \Psi_e^0, \Psi_e^1) | e \in \omega\}$ is an effective enumeration of all triples (Φ, Ψ^0, Ψ^1) such that every member of the triple is a p.c. functional. Clearly, meeting the requirements is sufficient to prove the theorem. For a p.c. functional, say Φ, which is not built by our construction, the use function φ will satisfy the following two properties:

(i) If $\varphi(x+1)[s]\downarrow$, then $\varphi(x)[s]\downarrow$ and $\varphi(x)[s] < \varphi(x+1)[s]$;

(ii) If $\varphi(x)[s]\downarrow$, then $x \leqslant \varphi(x)[s] < s$.

1.2 The $\mathcal{R}$-module

To satisfy $\mathcal{R}$, we will build a p.c. functional $\Omega(A_0, A_1)$. We first describe the properties of the use function ω, we call them ω-rules.

ω-rules:

ω1. If $\omega(k+1)[s] \neq \omega$, then $\omega(k)[s] < \omega(k+1)[s]$;

ω2. If $\Omega(A_0, A_1; x)[s]\uparrow$, then we set $\omega(x)[s] = \omega$; if $\omega(x)[s] \neq \omega$, then $\omega(x)[s] \notin A_{0,s} \cup A_{1,s}$;

ω3. If $\omega(x)[s] \neq \omega$ then $\omega(x)[s+1] = \omega$ if and only if $\omega(x)[s] \in (A_{0,s+1} \cup A_{1,s+1}) - (A_{0,s} \cup A_{1,s})$;

ω4. If $\omega(x)[s] \neq \omega = \omega(x)[s+1]$, then $(\forall y \geqslant x)[\omega(y)[s] \neq \omega \to \omega(y)[s] \in (A_{0,s+1} \cup A_{1,s+1})]$.

The $\mathcal{R}$-module will proceed as follows.

1. If $\Omega(A_0, A_1; k)\downarrow \neq K(k)$, then enumerate $\omega(x)$ for every $x \geqslant k$ with $\omega(x) \neq \omega$ into A_i for one and only one $i \in \{0, 1\}$;

2. Otherwise, let k be the least x such that $\omega(x) = \omega$. Define $\Omega(A_0, A_1; k) = K(k)$ with $\omega(k)$ fresh.

By the $\mathcal{R}$-module, for any k, $\lim_s \omega(k)\downarrow = \omega(k) < \omega$, and by the ω-rules, $\omega(k)$ is computable in $(A_0 \oplus A_1)$. For any fixed k, $(A_0 \oplus A_1)$-computably find the least stage s_k such that $\omega(k)[s_k] \notin A_0 \cup A_1$, then by the $\mathcal{R}$-module, $k \in K$ if and only if $k \in K_{s_k}$. $K \leqslant_T A_0 \oplus A_1$.

The problem is that the enumeration of ω-uses is not determined completely by the $\mathcal{R}$-module, that is to say, other strategies may also enumerate $\omega(k)$ for some k into $A_i (i = 0, 1)$. The point is that, for a fixed k, there are only finitely many strategies which may enumerate $\omega(k)$, and each of them will enumerate $\omega(k)$ only finitely many times. In this case, for any k, $\lim_s \omega(k)[s]\downarrow = \omega(k) < \omega$, and again $\omega(k)$ is computable in $A_0 \oplus A_1$. $K \leqslant_T A_0 \oplus A_1$. $\mathcal{R}$ is satisfied.

1.3 A $\mathcal{P}$-module

The basic module for a $\mathcal{P}$-requirement, say $\mathcal{P}_e^i$, is a standard Friedberg-Muchnik procedure. It will proceed as follows.

1. Appoint a witness, say y, which is a fresh odd number;

2. Wait for a stage at which $\Phi(A; y)\downarrow = 0 = A_{1-i}(y)$. Then enumerate y into A_{1-i} and preserve computation $\Phi(A; y)\downarrow = 0$.

1.4 An $\mathcal{M}$-module

For the sake of an $\mathcal{M}$-requirement, say $\mathcal{M}$ (we drop the index), we define the length function l of agreement between $\Psi^0(A_0, A)$ and $\Psi^1(A_1, A)$ as usual. We say that stage s is *$\mathcal{M}$-expansionary*, if $s = 0$ or the current

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length function $l>l[v]$ for all $v<s$. Clearly if there are only finitely many $\mathscr{M}$-expansionary stages, then $\mathscr{M}$ has already been satisfied. Therefore we consider only the case that there are infinitely many $\mathscr{M}$-expansionary stages. An $\mathscr{M}$-module will work only at $\mathscr{M}$-expansionary stages. An $\mathscr{M}$-module is the so-called Fejer's[13] method. It will build a p.c. functional $\Lambda(A)$ during the $\mathscr{M}$-expansionary stages and proceed as follows.

1. (Correct $\Lambda(A)$). If there is an x such that $\Lambda(A;x)\downarrow\neq\Psi^0(A_0,A;x)$ and $l>x$, then let p be the least such x, enumerate $\lambda(x)$ for every $x\geqslant p$ with $\lambda(x)\neq\omega$ into A;

2. (Build $\Lambda(A)$). Let p be the least x such that $\Lambda(A;x)\uparrow$. If $l>p$, then define $\Lambda(A;p)\downarrow=\Psi^0(A_0,A;p)$ with $\lambda(p)$ fresh.

For the use function λ of p.c. functional $\Lambda(A)$, we also require that the use function λ should satisfy the following properties. We call them *λ-rules*.

λ-rules:

(a) $\Lambda(A;x)\downarrow$ if and only if $\lambda(x)\neq\omega$;

(b) If $\Lambda(A;x+1)[s]\downarrow$, then $\Lambda(A;x)[s]\downarrow$ and $\lambda(x)[s]<\lambda(x+1)[s]$;

(c) If $\Lambda(A;x)[s]\downarrow$, then $\lambda(x)[s]\notin A_s$;

(d) If $\Lambda(A;x)[s]\downarrow$ then $\Lambda(A;x)[s+1]\uparrow$ if and only if $\lambda(x)[s]\in A_{s+1}-A_s$;

(e) If $\Lambda(A;x)[s]\downarrow$ and $\Lambda(A;x)[s+1]\uparrow$, then $(\forall y\geqslant x)[\lambda(y)[s]\neq=\omega\to\lambda(y)[s]\in A_{s+1}]$.

By the $\mathscr{M}$-module, if there are infinitely many $\mathscr{M}$-expansionary stages, then $\Lambda(A)$ is built infinitely often, and if $\Lambda(A)$ is total, then $\mathscr{M}$ is satisfied.

Clearly by the $\mathscr{M}$-module, if $\Psi^0(A_0,A)=\Psi^1(A_1,A)=g$ is total, then $\Lambda(A)$ is a total function. The problem is that the enumeration of λ-uses will not be determined completely by the $\mathscr{M}$-module, that is, other strategies may also enumerate λ-uses into A. The point is that for any fixed p, there are only finitely many strategies which may enumerate $\lambda(p)$, and each of them may enumerate $\lambda(p)$ only finitely many times unless some strategy has proven that there is an $x\leqslant p$ such that $\psi^i(x)$ will be unbounded for some $i\in\{0,1\}$

Thus in any case, $\mathscr{M}$ will be satisfied.

1.5 An $\mathscr{S}$-module

Suppose that δ is an $\mathscr{S}_e$-strategy. Then let $J^\delta=J^{\lfloor 2e\rfloor}$ and let $\omega^\delta=\omega^{\lfloor 2e\rfloor}$. δ will work with a *boundary* $b(\delta)$ which is defined by $b(\delta)=\max\{s\mid\delta$ is initialised at stage $s\}$. An $\mathscr{S}$-strategy will simply enumerate every $x>b(\delta)$, $x\in J^\delta-A$, into A. Clearly δ will satisfy its requirement unless $\lim_s b(\delta)[s]=\infty$, and in the latter case, we do not need this δ, since it is not on the true path.

1.6 A strategy below an $\mathscr{S}$-strategy

Suppose that ξ and δ are strategies, and that δ is an $\mathscr{S}$-strategy. Let $\delta\subset\xi$. If ξ assumes that δ has only finite actions, then δ injures ξ only finitely many times. Otherwise, ξ knows the computable set which is enumerated by δ. And then ξ can prevent the injury from δ. For example, ξ believes a computation $\Phi(A;y)\downarrow=z$, only if for every x, if $x>b(\delta)$, $x\in\omega^\delta$ and $x\leqslant\varphi(y)$, then x has already been enumerated into A. We now complete the description of the basic modules.

2 The Strategies

In this section, we will design the strategies for various combinations of the requirements.

2.1 The strategies for a triple $(\mathscr{S},\mathscr{M},\mathscr{P})$ of requirements $\mathscr{S},\mathscr{M}$ and $\mathscr{P}$

Suppose that δ,α and β are $\mathscr{S}$-, $\mathscr{M}$- and $\mathscr{P}$-strategies respectively. Let $\delta\subset\alpha\subset\beta$. β assumes that δ is enumerating an infinite computable set, and that α is building a p.c. functional $\Lambda(A)$. Suppose that β works on $\mathscr{P}_e^i$, then let $e(\beta)=e$ and $i(\beta)=i$. In the following discussion, we will drop the index $e(\beta)$.

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We say that $\Phi(A;y)\downarrow$ (or $\Psi^{i(\beta)}(A_{i(\beta)},A;p)\downarrow$) is *$\beta$-believable*, if for any x, if $x>b(\delta)$, $x\in\omega^{\delta}$ and $x\leqslant\varphi(y)$ (or $x\leqslant\psi^{i(\beta)}(p)$) then x has already been enumerated into A. The method to prevent the injury to a computation $\Phi(A;y)\downarrow=0$ from the building of the p.c. functional $\Lambda(A)$ is the Slaman's *configuration method*. β will impose an $A_{i(\beta)}$-restraint $r^{i(\beta)}(\beta)$. However, even if $r^{i(\beta)}(\beta)$ will not be injured by a strategy on the priority tree, it may be injured by the building of the p.c. functional $\Omega(A_0,A_1)$. The point is that β will work with a fixed *threshold*, say k. We will ensure that the $A_{i(\beta)}$-restraint $r^{i(\beta)}(\beta)$ will never be injured by an enumeration of $\omega(x)$ for any $x\geqslant k$. And if $r^{i(\beta)}(\beta)$ is injured by the building of $\Omega(A_0,A_1)$, then *reset* β, that is, cancel any previous progression of β but keep the threshold k unchanged. Thus if $r^{i(\beta)}(\beta)$ is injured by the enumeration of $\omega(x)$ for some x, then $x<k$. The point is that, for any x, $\omega(x)$ is enumerated only finitely many times, and then β is reset only finitely many times.

Thus the $\mathscr{P}$-strategy will proceed as follows.

1. Appoint a witness $y(\beta)$ to be a fresh odd number;
2. Wait for a stage, say s, at which $\Phi(A;y(\beta))\downarrow=0=A_{1-i(\beta)}(y(\beta))$ via β-believable computation. Run the following.

Program -1:

Step 1a. If $\varphi(y(\beta))>\varphi(y(\beta))[t^-,s^-]$, where $[t^-,s^-]$ is the greatest previous state at which step 2 occurred, then set $c(\beta)=\omega$;

[**Remark.** $c(\beta)$ is similar to a threshold of β for $\Lambda(A)$. We call it "*controller*" of β to distinguish it from the threshold $k=k(\beta)$.]

Step 1b. If $c(\beta)=\omega$, then define $c(\beta)$ as fresh;

Step 1c. If either $\lambda(c(\beta))\leqslant\psi^{i(\beta)}(c(\beta)-1)$ or there is an $x<c(\beta)$ such that $\Lambda(A;x)\neq\Psi^{i(\beta)}(A_{i(\beta)},A;x)$ or $\Psi^{i(\beta)}(A_{i(\beta)},A;x)$ is not β-believable, then enumerate $\lambda(x)$ for all $x\geqslant c(\beta)$ with $\lambda(x)\neq\omega$ into A;

Step 1d. Otherwise, enumerate $\lambda(x)$ for every $x\geqslant c(\beta)$ with $\lambda(x)\neq\omega$ into A, and go on to program 0.

Program 0: Let $r^{i(\beta)}(\beta)=\max\{\psi^{i(\beta)}(p)\mid p<c(\beta)\}$. There are two cases:

Case 0a. $\omega(k(\beta))\leqslant r^{i(\beta)}(\beta)$. Then enumerate $\omega(x)$ for every $x\geqslant k(\beta)$ with $\omega(x)\neq\omega$ into $A_{1-i(\beta)}$;

Case 0b. Otherwise, enumerate $y(\beta)$ into $A_{1-i(\beta)}$, enumerate $\omega(x)$ for every $x\geqslant k(\beta)$ with $\omega(x)\neq\omega$ into $A_{i(\beta)}$.

If Case 0b of program 0 occurs at a stage s_0 and $K\lceil k(\beta)=K_{s_0}\lceil k(\beta)$, then $\Phi(A;y(\beta))[s_0]\downarrow=0\neq1=A_{1-i(\beta)}(y(\beta))$ and $\Phi_\beta(A;y(\beta))[s_0]\downarrow=0$ will never be injured by the building of $\Lambda(A)$ at a stage $>s_0$. And then $\mathscr{P}_e^i$ is satisfied.

If Case 0a of program 0 occurs at a stage s_0 and $K\lceil k(\beta)=K_{s_0}\lceil k(\beta)$, then Case 0b of Step 0 will occur at the next α-expansionary stage, say s_1, and then $\Phi_\beta(A;y(\beta))\downarrow=0$ will be preserved and $y(\beta)$ will be enumerated into $A_{1-i(\beta)}$. $\Phi_\beta(A;y(\beta))\downarrow=0\neq1=A_{1-i(\beta)}(y(\beta))$. $\mathscr{P}_e^i$ is satisfied.

If $\lim_s(y(\beta))[s]=\infty$, then $\Phi(A;y(\beta))$ diverges, and then $\mathscr{P}_e^i$ is satisfied. And in this case, $\lim_s c(\beta)[s]=\infty$, and then for any fixed p, β enumerates $\lambda(p)$ only finitely many times. $\Lambda(A)$ is still correct. If Step 2 occurs only finitely many times, then either $\varphi(y(\beta))$ will be unbounded or $\Phi(A;y(\beta))\neq0=A_{1-i(\beta)}(y(\beta))$. β satisfies its $\mathscr{P}$-requirement.

Otherwise, $\lim_s c(\beta)[s]\downarrow=c(\beta)<\omega$. And then for almost every stage at which Step 2 occurs, Step 1c of program -1 will occur. By the strategy, there is a $p<c(\beta)$ such that $\psi^{i(\beta)}(p)$ will be unbounded and $\lambda(c(\beta))$ will be unbounded. Therefore β proves that $\mathscr{M}$ has already been satisfied, but β fails to satisfy its $\mathscr{P}$-requirement. In this case, there are finitely many *backup strategies* S^p for every $p\leqslant c(\beta)$. S^p assumes that p is the least x such that $\lambda(x)$ will be unbounded. And the $\mathscr{P}$-requirement of β will be re-arranged to a backup strategy for the strategy β, say S^p. The point is that the backup strategy S^p will never be injured by the building

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of $\Lambda(A)$.

2.2 The strategies for all requirements below one $\mathscr{M}$-strategy

An $\mathscr{I}$-strategy will be the same as an $\mathscr{I}$-module. Suppose that α is an $\mathscr{M}$-strategy. If there is a $\mathscr{P}$-strategy β such that $\alpha\subset\beta$ and β fails to satisfy its $\mathscr{P}$-requirement, then β proves that the $\mathscr{M}$-requirement of α has already been satisfied. And then all other requirements will be satisfied by a backup strategy S^p for the β for some p.

Otherwise. Every $\mathscr{P}$-requirement will be satisfied by a $\mathscr{P}$-strategy $\beta\supset\alpha$. And clearly, every $\mathscr{I}$-requirement will be satisfied by an $\mathscr{I}$-strategy $\delta\supset\alpha$. And by the $\mathscr{M}$-strategy α, $\Lambda(A)$ will be built infinitely often. By the definition of $c(\beta)$ for the $\mathscr{P}$-strategy β, for any fixed p, p will be defined to be $c(\beta)$ for some $\mathscr{P}$-strategy β at most once. And by the assumption of this case, for every $\beta\supset\alpha$, either β has only finite actions or $\lim_s c(\beta)=\infty$. Therefore $\lambda(p)$ will be enumerated by $\mathscr{P}$-strategies only finitely many times. Thus $\Lambda(A)$ will be total unless there is a fixed p such that $\lambda(p)$ will be enumerated infinitely many times by $\mathscr{M}$-strategy α itself. By the $\mathscr{M}$-strategy, in the latter case, either $\psi^0(p)$ or $\psi^1(p)$ will be unbounded. Thus in any case, $\mathscr{M}$ is satisfied. Therefore in any case, $\mathscr{M}$ and all other requirements will eventually be satisfied.

2.3 A $\mathscr{P}$-strategy below finitely many $\mathscr{M}$-strategies

Generally, a $\mathscr{P}$-strategy β will satisfy its $\mathscr{P}$-requirement while priority is given to a finite number of $\mathscr{M}$-strategies which are building p.c. functionals $\Lambda_j(A)$ for $j=1,2,\ldots,m$.

β will always assume that, for any p, $\lambda_1(p)<\lambda_2(p)<\ldots<\lambda_m(p)$ (if they are defined). As in section 2.1, β will define a finite number of parameters:

- $y(\beta)$: the current witness of β;
- $c(\beta,j)$: the controller of β for $\Lambda_j(A)$, $j\in\{1,2,\ldots,m\}$;
- $r^{i(\beta)}(\beta)$: the $A_{i(\beta)}$-restraint of β;
- $u(\beta)$: $\max\{\varphi_\beta(y(\beta)), r^{i(\beta)}(\beta)\}$.

β will proceed as follows.

β1 Appoint witness $y(\beta)$ to be a fresh odd number;

β2 Wait for a stage, say s, at which $\Phi_\beta(A;y(\beta))\downarrow=0=A_{1-i(\beta)}(y(\beta))$ via β-believable computation. Execute program $-m$.

Program $-m$:

Step ma. If $\varphi(y(\beta))>\varphi(y(\beta))[t^-,s^-]$, where $[t^-,s^-]$ is the greatest previous state at which β2 occurred, then set $c(\beta,j)=\omega$ for every $j\in\{1,2,\ldots,m\}$;

Step mb. If $c(\beta,m)=\omega$, then define $c(\beta,m)$ to be a fresh number;

Step mc. If either $\lambda_m(c(\beta,m))\leqslant\psi_m^{i(\beta)}(c(\beta,m)-1)$ or there is an $x<c(\beta,m)$ such that either $\Lambda_m(A;x)\neq\Psi_m^{i(\beta)}(A_{i(\beta)},A;x)$ or $\Psi_m^{i(\beta)}(A_{i(\beta)},A;x)$ is not β-believable, then enumerate $\lambda_m(x)$ for every $x\geqslant c(\beta,m)$ with $\lambda_m(x)\neq\omega$ into A;

Step md. Otherwise, enumerate $\lambda_m(x)$ for every $x\geqslant c(\beta,m)$ with $\lambda_m(x)\neq\omega$ into A and go on to program $-m+1$;

...

Program $-j$:

Step ja. If $c(\beta,j)$ was defined at substage $t+1$ of stage s^- for some t, and $\psi_{j+1}^{i(\beta)}(c(\beta,j+1)-1)\neq\psi_{j+1}^{i(\beta)}(c(\beta,j+1)-1)[t,s^-]$, then set $c(\beta,i)=\omega$ for every $i\in\{1,2,\ldots,j\}$;

Step jb. If $c(\beta,j)=\omega$, then define $c(\beta,j)$ as fresh;

Step jc. If either $\lambda_j(c(\beta,j))\leqslant\psi_j^{i(\beta)}(c(\beta,j)-1)$ or there is an $x<c(\beta,j)$ such that either $\Lambda_j(A;x)\neq\Psi_j^{i(\beta)}$

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$(A_{i(\beta)}, A; x)$ or $\Psi_j^{i(\beta)}(A_{i(\beta)}, A; x)$ is not β-believable, then enumerate $\lambda_i(x)$ for each $i \in \{j, j+1, \ldots, m\}$ and every $x \geq c(\beta, j)$ with $\lambda_i(x) \neq \omega$ into A;

Step jd. Otherwise, enumerate $\lambda_j(x)$ for every $x \geq c(\beta, j)$ with $\lambda_j(x) \neq \omega$ into A and go on to program $-j+1$;

...

Program -1:

Step 1a. If $c(\beta,1)$ was defined at substage $t+1$ of stage s^- for some t and $\psi_2^{i(\beta)}(c(\beta,2)-1) \neq \psi_2^{i(\beta)}(c(\beta,2)-1)[t, s^-]$, then set $c(\beta,1)=\omega$;

Step 1b. If $c(\beta,1)=\omega$, then define $c(\beta,1)$ as fresh;

Step 1c. If either $\lambda_1(c(\beta,1)) \leq \psi_1^{i(\beta)}(c(\beta,1)-1)$ or there is an $x < c(\beta,1)$ such that either $A_i(A; x) \neq \Psi_1^{i(\beta)}(A_{i(\beta)}, A; x)$ or $\Psi_1^{i(\beta)}(A_{i(\beta)}, A; x)$ is not β-believable, then enumerate $\lambda_i(x)$ for each $i \in \{1, 2, \ldots, m\}$ and for every $x \geq c(\beta,1)$ with $\lambda_i(x) \neq \omega$ into A;

Step 1d. Otherwise, enumerate $\lambda_1(x)$ for every $x \geq c(\beta,1)$ with $\lambda(x) \neq \omega$ into A and go on to program 0;

Program 0: Let $r^{i(\beta)}(\beta) = \max\{\psi_j^{i(\beta)}(p) \mid p < c(\beta, j),\ j = 1, 2, \ldots, m\}$, and let $u(\beta) = \max\{r^{i(\beta)}(\beta), \varphi_\beta(y(\beta))\}$. There are two cases:

Case 0a. $\omega(k(\beta)) \leq r^{i(\beta)}(\beta)$, then enumerate $\omega(x)$ for every $x \geq k(\beta)$ with $\omega(x) \neq \omega$ into $A_{1-i(\beta)}$;

Case 0b. Otherwise, enumerate $y(\beta)$ into $A_{1-i(\beta)}$, enumerate $\omega(x)$ for every $x \geq k(\beta)$ with $\omega(x) \neq \omega$ into $A_{i(\beta)}$.

2.4 The possible outcomes of the $\mathscr{P}$-strategy

If program 0 occurs at a stage s_0 and $\omega(k)[s_0]\downarrow \notin A_0 \cup A_1$ for every $k < k(\beta)$, then $\Phi(A; y(\beta))[s_0]\downarrow = 0$ and it will never be injured by the building of $A_j(A)$ for any $j \in \{1, 2, \ldots, m\}$. And then $\Phi(A; y(\beta))[s_0]\downarrow = 0$ will be preserved forever. If Case 0b of program 0 occurs at stage s_0, then $A_{1-i(\beta)}(y(\beta))[s_0]=1$. If Case 0a of program 0 occurs at stage s_0, then $y(\beta)$ will be enumerated into $A_{1-i(\beta)}$ at the next stage $s_1 > s_0$ at which β is visited. Thus if program 0 occurs at a stage s_0, and for any $k < k(\beta)$, $\omega(k)[s_0]\downarrow \notin A_0 \cup A_1$, $\Phi(A; y(\beta))\downarrow = 0 \neq 1 = A_{1-i(\beta)}(y(\beta))$. $\mathscr{P}$ is satisfied.

If $\beta 2$ occurs only finitely many times and program 0 will not hold permanently, then either $\varphi_\beta(y(\beta))$ will be unbounded or $\Phi_\beta(A; y(\beta)) \neq 0 = A_{1-i(\beta)}(y(\beta))$. In either case, $\mathscr{P}$ is satisfied.

If $\lim_s c(\beta, m)[s] = \infty$, then $\varphi(y(\beta))$ will be unbounded, $\mathscr{P}$ is satisfied and by the definition of $c(\beta, j)$, for every $j \in \{1, 2, \ldots, m\}$, $\lim_s c(\beta, j)[s] = \infty$. $A_j(A)$ is still correct for every $j \in \{1, 2, \ldots, m\}$.

Otherwise, $\lim_s c(\beta, m)[s]\downarrow = c(\beta, m) < \omega$. Let j be the least i such that $\lim_s c(\beta, i)[s]\downarrow = c(\beta, i) < \omega$. By the strategy, $\psi_j^{i(\beta)}(c(\beta, j))$ will be unbounded, and $\lambda_i(c(\beta, j))$ will be unbounded for every $i \in \{j, j+1, \ldots, m\}$. $\mathscr{M}_j$ has been satisfied, and for any $i \in \{1, 2, \ldots, j-1\}$, $A_i(A)$ is still correct. In this case, there are finitely many backup strategies below β. A backup strategy of β will guess for each $i \in \{j, j+1, \ldots, m\}$ the least p such that $\lambda_i(p)$ will be unbounded, and then the backup strategy will not be injured by the building of $A_i(A)$ for any $i \in \{j, j+1, \ldots, m\}$.

2.5 A backup strategy for the $\mathscr{P}$-strategy β

Suppose that j is the least i such that $\lim_s c(\beta, i)[s]\downarrow = c(\beta, i) < \omega$. Let $\tau = \langle(2, -j)\rangle \hat{}\ \langle(f_j, p_j)\rangle \hat{}\ \langle(f_{j+1}, p_{j+1})\rangle \hat{}\ \ldots \hat{}\ \langle(f_m, p_m)\rangle$, such that $p_j \geq p_{j+1} \geq \ldots\ p_m$. Let $\beta^* = \beta \hat{}\ \tau$. Then β^* is a backup strategy of β, and β^* believes that p_i is the least p such that $\lambda_i(p)$ will be unbounded for every $i \in \{j, j+1, \ldots, m\}$. The $\mathscr{P}$-requirement of β will be rearranged to nodes $\supset \beta^*$. The backup strategy deals with the injury from at most $m-1$ p.c. functionals $A_1(A), \ldots, A_{j-1}(A), A'_{j+1}(A), \ldots, A'_m(A)$. Then a backup strategy of β is similar to and simpler than the general $\mathscr{P}$-strategy β in section 2.3.

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3 The Priority Tree T

In this section, we will describe the building of the priority tree. The priority tree will grow upwardly. For some node α, there is a number $e(\alpha)$ or a pair $\langle e(\alpha),i(\alpha)\rangle$ of numbers which is associated with it naturally. Intuitively, it is the index of the requirement on which α works. For example, if $e(\alpha)\downarrow$ and $i(\alpha)\downarrow$, then α is a $\mathscr{P}$-strategy which works on $\mathscr{P}_e^i$ for $e=e(\alpha)$ and $i=i(\alpha)$. For some node α, we define neither $e(\alpha)$ nor $i(\alpha)$. Intuitively, α is a *real strategy* if and only if $e(\alpha)\downarrow$. And if $e(\alpha)\uparrow$, then we call α a *virtual strategy*. [The formal definition of the real and the virtual strategies will be given in definition 3.2.]

The priority tree T will be built by an effective construction which enumerates the nodes of T. If $e(\alpha)$ (or $\langle e(\alpha),i(\alpha)\rangle$) is defined then it will be defined at the stage at which α is enumerated into T.

To build the priority tree T, we first give some notations.

Definition 3.1. (i) Define the priority ranking of the requirements by $o(\mathscr{R})=-1$, $o(\mathscr{M}_e)=4e$, $o(\mathscr{S}_e)=4e+1$, $o(\mathscr{P}_e^0)=4e+2$ and $o(\mathscr{P}_e^1)=4e+3$ for all $e\in\omega$.

We define $\mathscr{X}<\mathscr{Y}$ for requirements $\mathscr{X}$ and $\mathscr{Y}$, if $o(\mathscr{X})<o(\mathscr{Y})$.

Give a node $\xi\in T$.

(ii) We say that γ *has been destroyed at* ξ if $\gamma\subset\xi$ and there exists an α and a β such that $\alpha\hat{}\langle(0,e(\alpha))\rangle\subseteq\gamma\subset\beta\hat{}\langle(2,-j)\rangle\hat{}\langle(f_\alpha,p)\rangle\subseteq\xi$ for some $j>0$ and for some $p\in\omega$; and we say that γ is correct at ξ if $\gamma\subset\xi$ and γ has not been destroyed at ξ yet.

(iii) A real $\mathscr{S}$-strategy δ with $\delta\subset\xi$ has been satisfied at ξ if δ is correct at ξ.

(iv) A real $\mathscr{M}$-strategy α with $\alpha\subset\xi$ has been satisfied at ξ, if α is correct at ξ and $\alpha\hat{}\langle(0,e(\alpha))\rangle\subseteq\beta\subset\beta\hat{}\langle(2,-j)\rangle\hat{}\langle(f_\alpha,p)\rangle\subseteq\xi$ for some $j>0$ and for some $p\in\omega$.

(v) A real $\mathscr{M}$-strategy α with $\alpha\subset\xi$ is active at ξ if α is correct at ξ and α has not been satisfied at ξ.

(vi) A real $\mathscr{P}$-strategy β with $\beta\subset\xi$ has been satisfied at ξ if β is correct at ξ and $\beta\hat{}\langle a\rangle\subseteq\xi$, $a=(n,\langle e(\beta),i(\beta)\rangle)$ for some $n=0,1$ or 3.

(vii) $\mathscr{S}_e$ or $\mathscr{M}_e$ has been satisfied at ξ if there is a real $\mathscr{S}_e$- or a real $\mathscr{M}_e$-strategy which has been satisfied at ξ.

(viii) $\mathscr{P}_e^i$ has been satisfied at ξ if there is a real $\mathscr{P}_e^i$-strategy β such that β has been satisfied at ξ.

(ix) $\mathscr{M}_e$ is active at ξ if $\mathscr{M}_e$ has not been satisfied at ξ and there is a real $\mathscr{M}_e$-strategy α such that α is active at ξ.

We now define the possible outcomes of the strategies.

Definition 3.2. Given a real $\mathscr{P}$-strategy β, let $e=\langle e(\beta),i(\beta)\rangle$. Suppose that all real $\mathscr{M}$-strategies which are active at β are $\alpha_1,\alpha_2,\ldots,\alpha_m$ with $\alpha_1\subset\alpha_2\subset\ldots\subset\alpha_m\subset\beta$. Then

(i) Define $m(\beta)=m$;

(ii) Define the possible outcomes of β as follows: $\langle(0,e)\rangle,\langle(1,e)\rangle,\langle(3,e)\rangle$ and $\langle(2,-j)\rangle\hat{}\langle(f_{\alpha_j},p_j)\rangle\hat{}\langle(f_{\alpha_{j+1}},p_{j+1})\rangle\hat{}\ldots\hat{}\langle(f_{\alpha_m},p_m)\rangle$ for $m=m(\beta)$, $p_j\geqslant p_{j+1}\geqslant\ldots\geqslant p_m$, and $j\in\{1,2,\ldots,m(\beta)\}$;

(iii) For any $\gamma=\beta\hat{}\langle(2,-j)\rangle\hat{}\langle(f_{\alpha_j},p_j)\rangle\hat{}\langle(f_{\alpha_{j+1}},p_{j+1})\rangle\hat{}\ldots\hat{}\langle(f_{\alpha_m},p_m)\rangle$ for $m=m(\beta)$ with $p_j\geqslant p_{j+1}\geqslant\ldots\geqslant p_m$, γ is a *real strategy* and every ξ with $\beta\subset\xi\subset\gamma$ is a *virtual strategy*, and every $\gamma=\beta\hat{}\langle a\rangle$ for $a\in\{(0,e),(1,e),(3,e)\}$ is a *real strategy*.

Suppose that α is a real $\mathscr{S}$ or $\mathscr{M}$-strategy. Then

(iv) The possible outcomes of α are $\langle(0,e(\alpha))\rangle$ and $\langle(1,e(\alpha))\rangle$, and both $\alpha\hat{}\langle(0,e(\alpha))\rangle$ and $\alpha\hat{}\langle(1,e(\alpha))\rangle$ are *real strategies*;

(v) We define the ordering of the possible outcomes as the usual lexicographical order.

We now describe the priority tree T.

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Definition 3.3. (i) We define the root node $\lambda=\langle(-1,1)\rangle$ as a real $\mathscr{M}_0$-strategy;

(ii) The immediate successors of a real strategy are the possible outcomes of the corresponding strategy as described in definition 3.2;

(iii) Suppose that we have enumerated ξ to be a real strategy but $e(\xi)$ has not been defined, then find the requirement $\mathscr{R}$ with $o(\mathscr{R})$ minimal such that $\mathscr{R}$ has not been satisfied at ξ and $\mathscr{R}$ is not active at ξ; if $\mathscr{R}$ is $\mathscr{I}_e$ or $\mathscr{M}_e$ then define ξ to be a real $\mathscr{I}$ or $\mathscr{M}$-strategy respectively with $e(\xi)=e$ and if $\mathscr{R}=\mathscr{P}_e^i$ for some e, i, then we say that ξ is a real $\mathscr{P}$-strategy with $e(\xi)=e$ and $i(\xi)=i$.

Then the priority tree T is built as follows.

Definition 3.4. (i) The priority tree T will be built as a c.e. set of strategies while the full construction proceeds;

(ii) At the state at which a strategy ξ appears in the full construction for the first time, we enumerate ξ into T simultaneously and automatically;

(iii) If a real strategy ξ is in T, and $e(\xi)$ has not been defined, then define $e(\xi)$ or $\langle e(\xi), i(\xi)\rangle$ by definition 3.3 simultaneously and automatically;

(iv) At any point during the full construction, we always respect the lexicographical ordering of nodes.

We now have built the priority tree T to be a computably enumerable set of all strategies (either real or virtual) which appear in the full construction.

4 The Full Construction

Before describing the construction, we first give some notations and parameters.

Definition 4.1. (i) For a real $\mathscr{M}$-strategy α, we say that s is α-expansionary, if $s=0$ or $l(\alpha)>l(\alpha)[t]$ for all $t<s$ at which α is visited, where $l(\alpha)$ is the current length of agreement between $\Psi_\alpha^0(A_0,A)$ and $\Psi_\alpha^1(A_1,A)$, and $\Psi_\alpha^i(A_i,A)$ is the version of $\Psi_{e(\alpha)}^i(A_i,A)$ which is computed by α;

(ii) We fix an infinite partition $\bigcup_{e\in\omega}\omega^{[2e+1]}=\{D_0<D_1<D_2<\ldots\}$ such that every D_i is an infinite computable set;

(iii) For a real $\mathscr{I}$-strategy δ, let $e=e(\delta)$, then define $J^\delta(=J^{[2e]})$ to be the version of $J^{[2e]}$ which is enumerated by δ, and define $\omega^\delta=\omega^{[2e]}$;

(iv) Given a real $\mathscr{P}$-strategy β, suppose that all real $\mathscr{M}$-strategies which are active at β are $\alpha_1,\alpha_2,\ldots,\alpha_m$ with $\alpha_1\subset\alpha_2\subset\ldots\subset\alpha_m$. Then

(a) Define $m(\beta)=m$;

(b) If $m(\beta)=0$, then β has only one parameter $y(\beta)$ (the current witness of β);

(c) If $m(\beta)\neq 0$, then β will have the following parameters:

- $k(\beta)$: the threshold of β;
- $y(\beta)$: the current witness of β;
- $c(\beta,j)$: the controllers of β, where $j\in\{1,2,\ldots,m(\beta)\}$;
- $r^{i(\beta)}(\beta)$: the $A_{i(\beta)}$-restraint which is imposed by β; and
- $u(\beta)$: $\max\{\varphi_\beta(y(\beta)), r^{i(\beta)}(\beta)\}$.

Definition 4.2. Suppose that β is a real $\mathscr{P}$-strategy. We say that $\Phi(A;y(\beta))\downarrow=0$ (or $\Psi^{i(\beta)}(A_{i(\beta)},A;p)\downarrow$) is β-believable, if both (i) and (ii) below hold:

(i) For any $\mathscr{I}$-strategy δ with $\delta^\frown\langle(0,e(\delta))\rangle\subseteq\beta$, every x with $x>b(\delta)$, $x\in\omega^\delta$ and $x\leqslant\varphi(y(\beta))$ (or $x\leqslant\psi^{i(\beta)}(p)$) has already been enumerated into A;

(ii) For any ξ such that $\xi^\frown\langle(f_\alpha,p)\rangle\subseteq\beta$ for some α and some p, $\lambda_\alpha(p)>\varphi(y(\beta))$ (or $\lambda_\alpha(p)>\psi^{i(\beta)}(p)$).

For a p.c. functional which is built by our construction, we always assume that the use function will have

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some properties.

Definition 4.3. (i) For the p.c. functional Ω, we assume the ω-rules ω1~ω4 in 1.2;

(ii) For a p.c. functional Λ, we assume the λ-rules (a)~(e) in 1.4.

During the course of a construction, we often *initialise some strategies*. We now define the action of initialisation.

Definition 4.4 (Initialisation). (i) If a real $\mathscr{P}$-strategy β is initialised, then set $k(\beta)=y(\beta)=r^{i(\beta)}(\beta)=-1$ and set $c(\beta,j)=\omega$ for every $j\in\{1,2,\ldots,c(\beta)\}$;

(ii) If a real $\mathscr{P}$-strategy β is reset, then set $y(\beta)=r^{i(\beta)}(\beta)=-1$ and set $c(\beta,j)=\omega$ for every $j\in\{1,2,\ldots,c(\beta)\}$;

(iii) If a real $\mathscr{M}$-strategy α is initialised, then set $\Lambda_\alpha(A)$ to be totally undefined without any enumeration;

(iv) If a real strategy ξ is initialised, then any link (α,ξ) or (ξ,β) for any α and any β will be cancelled.

For the convenience of the description of the full construction, we define the notion of the ENUMERATE.

Definition 4.5. (i) We say that ENUMERATE $\omega(x)$ into A_i $(i=0,1)$ if we enumerate $\omega(y)$ for every $y\geqslant x$ with $\omega(y)\neq\omega$ into A_i;

(ii) For a real $\mathscr{M}$ strategy α, and for a natural number p, we define the notion ENUMERATE $\lambda_\alpha(p)$ by the following actions:

(a) Enumerate $\lambda_\alpha(x)$ for every $x\geqslant p$ with $\lambda_\alpha(x)\neq\omega$ into A;

(b) For any real $\mathscr{M}$-strategy α', if $\alpha\subset\alpha'$ and α is active at α', then enumerate $\lambda_{\alpha'}(x)$ for every $x\geqslant p$ with $\lambda_{\alpha'}(x)\neq\omega$ into A.

During the course of the construction, there are a number of actions which will be executed automatically. We list them as follows, and we will not mention them in the description of the full construction.

Definition 4.6 (Automatic Action). (i) If ξ is initialised or reset, then any γ with $\xi\subset\gamma$ is initialised simultaneously and automatically;

(ii) If a strategy (either real or virtual strategy) appears in the construction, and ξ has not been enumerated into T, then enumerate ξ into T immediately and automatically;

(iii) If a real strategy ξ is in T, and $e(\xi)$ has not been defined, then define $e(\xi)$ or $\langle e(\xi),i(\xi)\rangle$ by definition 3.3 (iii) immediately and automatically;

(iv) If a real $\mathscr{M}$-strategy α is in T, $e(\alpha)$ is defined and D_α has not been defined, then define D_α immediately and automatically such that D_α is the $<$-least element D_i which has not been used;

(v) If a real $\mathscr{S}$-strategy δ is initialised, then redefine $b(\delta)$ to be the current stage immediately and automatically.

We note that there is no action which is taken for a node ξ until ξ is enumerated into T. The full construction will be divided into odd stages and even stages. We fix an enumeration $\{K_s|s\in\omega\}$ of K, without repetition such that $K_0=\varnothing$, $K_s=K_{s-1}$ if s is even, and $|K_s-K_{s-1}|=1$, if s is odd. We now describe the full construction.

Definition 4.7 (The Construction). Stage $s=0$. Set $T=A=A_0=A_1=\varnothing$; Stage $s=2n+1$. Let $k_s\in K_s-K_{s-1}$. There are two cases:

Case 1. $\omega(k_s)\neq\omega$.

(i) If there is a $\mathscr{P}$-strategy ξ such that $\omega(k_s)\leqslant r^{i(\xi)}(\xi)$, then let β be the unique ξ with $k(\xi)$ minimal (see Proposition 5.1 (iii)). Then

- We say that β receives attention at stage s;
- ENUMERATE $\omega(k_s)$ in $A_{1-i(\beta)}$;
- Reset any γ with $k(\beta)<k(\gamma)$.

(ii) Otherwise, then ENUMERATE $\omega(k_s)$ in A_0, in either subcase, go to stage $s+1$.

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Case 2. Otherwise, then let k be the least x such that $\omega(x)=\omega$. Define $\omega(k)$ to be a fresh even number, and go to stage $s+1$.

Stage $s=2(n+1)$. We say that ξ is *visited* at stage s, if ξ is *eligible to act* at a substage of stage s. We first allow the root node to be eligible to act at substage 0.

Substage t. Let ξ be eligible to act at substage t of stage s. If $t=s$, then initialise any γ with $\xi<_L\gamma$ and go to stage $s+1$; if $t<s$ and ξ is a real strategy, then there are three cases, otherwise, go to the action phase of stage s.

Case A. $\xi=\alpha$ is an $\mathscr{M}$-strategy. Then run program α below.

Program α:

$\alpha1$. If s is not α-expansionary, then let $\alpha\hat{\ }\langle(1,e(\alpha))\rangle$ be eligible to act next (i.e., $\alpha\hat{\ }\langle(1,e(\alpha))\rangle$ is eligible to act at substage $t-1$ of stage s);

$\alpha2$. If there is a link (α,ξ) which was created and which has neither been travelled nor been cancelled for some ξ, then let β be the $<$-least such ξ, and go to program β (in Case C below);

$\alpha3$. If there is an x such that $\Lambda_\alpha(A;x)\downarrow$, $l(\alpha)>x$ and $\Lambda_\alpha(A;x)\neq\Psi_\alpha^0(A_0,A;x)$, then let p be the least such x, ENUMERATE $\lambda_\alpha(p)$, initialise any γ with $\alpha\hat{\ }\langle(0,e(\alpha))\rangle<_L\gamma$ and go to stage $s+1$;

$\alpha4$. If there is an x such that $\lambda_\alpha(x)\leqslant\min\{\psi_\alpha^0(x),\psi_\alpha^1(x)\}$, then let p be the least such x, ENUMERATE $\lambda_\alpha(p)$, initialise any γ with $\alpha\hat{\ }\langle(0,e(\alpha))\rangle<_L\gamma$ and go to stage $s+1$;

$\alpha5$. Let p be the least x such that $\lambda_\alpha(x)=\omega$. If for any $\alpha'\subset\alpha$ such that α' is active at α, $\lambda_{\alpha'}(p)\neq\omega$, and $l(\alpha)>p$, then define $\Lambda_\alpha(A;p)=\Psi_\alpha^0(A_0,A;p)$ and define $\lambda_\alpha(p)$ to be the fresh $y\in D_\alpha$, and let $\alpha\hat{\ }\langle(0,e(\alpha))\rangle$ be eligible to act next;

$\alpha6$. Otherwise, initialise any γ with $\alpha<_L\gamma$ and go to stage $s+1$.

Case B. $\xi=\delta$ is an $\mathscr{S}$-strategy. Then run program δ below.

Program δ:

$\delta1$. If there is an x such that $x>b(\delta)$ and $x\in J^\delta-A$, then enumerate x into A, and let $\delta\hat{\ }\langle(0,e(\delta))\rangle$ be eligible to act next;

$\delta2$. Otherwise, let $\delta\hat{\ }\langle(1,e(\delta))\rangle$ be eligible to act next.

Case C. $\xi=\beta$ is a $\mathscr{P}$-strategy. Then run program β below.

Program β:

$\beta1$. If $\Phi_\beta(A;y(\beta))\downarrow=0\neq1=A_{1-i(\beta)}(y(\beta))$ has been created since β was initialised or reset for the last time, then let $\beta\hat{\ }\langle(0,\langle e(\beta),i(\beta)\rangle)\rangle$ be eligible to act next.

$\beta2$. (i) If there is a link (α,β) (for some real $\mathscr{M}$-strategy α) which was created and which has neither been travelled nor been cancelled, then travel this link by case 0b of program 0 of $\beta3$ below;

(ii) Otherwise, go to $\beta3$.

$\beta3$. Suppose that $\Phi_\beta(A;y(\beta))\downarrow=0=A_{1-i(\beta)}(y(\beta))$ via β-believable computation. If $m(\beta)=0$, then enumerate $y(\beta)$ into $A_{1-i(\beta)}$, initialise any $\gamma\not\leqslant\beta$ and go to stage $s+1$.

Otherwise, let $m=m(\beta)$, and go to program $-m$ below.

Program $-m$:

Step ma. If $c(\beta,m)$ was defined at state $[t^-,s^-]$, and $\varphi_\beta(y(\beta))>\varphi_\beta(y(\beta))[t^-,s^-]$, then set $c(\beta,j)=\omega$ for every $j\in\{1,2,\ldots,m(\beta)\}$, and let $\beta\hat{\ }\langle(1,e)\rangle$ (where $e=\langle e(\beta),i(\beta)\rangle$) be eligible to act next;

Step mb. If $c(\beta,m)=\omega$, then define $c(\beta,m)$ to be fresh, and set $c(\beta,j)=\omega$ for every $j\in\{1,2,\ldots,m(\beta)-1\}$;

Step mc. If either $\lambda_{\alpha_m}(c(\beta,m))\leqslant\psi_{\alpha_m}^{i(\beta)}(c(\beta,m)-1)$ or there is an $x<c(\beta,m)$ such that either $\Lambda_{\alpha_m}(A;x)\neq\Psi_{\alpha_m}^{i(\beta)}(A_{i(\beta)},A;x)$ or $\Psi_{\alpha_m}^{i(\beta)}(A_{i(\beta)},A;x)$ is not β-believable, then

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- ENUMERATE $\lambda_{\alpha_m}(c(\beta,m))$;
- Set $c(\beta,j)=\omega$ for every $j\in\{1,2,\ldots,m(\beta)-1\}$;
- For every $x\leqslant c(\beta,m)$, define s_x to be the greatest stage $s'<s$ at which $\beta^\frown\langle(2,-m)\rangle^\frown\langle(f_{\alpha_m},x)\rangle$ was either visited or initialised, if $\beta^\frown\langle(2,-m)\rangle^\frown\langle(f_{\alpha_m},x)\rangle$ was visited or initialised, and let s_x be the greatest stage $s'<s$ at which β was visited, otherwise; let p be the least x such that $\max\{\lambda_{\alpha_m}(x)[t]\neq\omega\mid s_x\leqslant t<s\}>\max\{\lambda_{\alpha_m}(x)[t]\neq\omega\mid t<s_x\}$; let $\beta^\frown\langle(2,-m)\rangle^\frown\langle(f_{\alpha_m},p)\rangle$ be eligible to act next;

Step md. Otherwise, ENUMERATE $\lambda_{\alpha_m}(c(\beta,m))$ and go on to program $-m+1$ below

....

Program $-j$:

Step ja. If $c(\beta,j)$ was defined at state $[t^-,s^-]$, and $\psi^{i(\beta)}_{\alpha_{j+1}}(c(\beta,j+1)-1)\neq\psi^{i(\beta)}_{\alpha_{j+1}}(c(\beta,j+1)-1)[t^-,s^-]$, then set $c(\beta,i)=\omega$ for every $i\in\{1,2,\ldots,j\}$;

Step jb. If $c(\beta,j)=\omega$, then define $c(\beta,j)$ to be fresh;

Step jc. If either $\lambda_{\alpha_j}\leqslant\psi^{i(\beta)}_{\alpha_j}(c(\beta,j)-1)$ or there is an $x<c(\beta,j)$ such that either $\Lambda_{\alpha_j}(A;x)\neq\Psi^{i(\beta)}_{\alpha_j}(A_{i(\beta)},A;x)$ or $\Psi^{i(\beta)}_{\alpha_j}(A_{i(\beta)},A;x)$ is not β-believable, then

- ENUMERATE $\lambda_{\alpha_j}(c(\beta,j))$;
- Set $c(\beta,i)=\omega$ for every $i\in\{1,2,\ldots,j-1\}$;
- For every $x\leqslant c(\beta,j)$, define s_x to be the greatest stage $s'<s$ at which $\beta^\frown\langle(2,-j)\rangle^\frown\langle(f_{\alpha_j},x)\rangle$ was either visited or initialised, if $\beta^\frown\langle(2,-j)\rangle^\frown\langle(f(\alpha_j,x)\rangle$ was visited or initialised, and define s_x to be the greatest stage $s'<s$ at which β was visited, otherwise; let p be the least x such that $\max\{\lambda_{\alpha_j}(x)[t]\neq\omega\mid s_x\leqslant t<s\}>\max\{\lambda_{\alpha_j}(x)[t]\neq\omega\mid t<s_x\}$; let $\beta^\frown\langle(2,-j)\rangle^\frown\langle(f_{\alpha_j},p)\rangle$ be eligible to act next;

Step jd. Otherwise, ENUMERATE $f_{\alpha_j}(n(\beta,j))$ and go on to program $-j+1$.

...

Program 0: Let $r^{i(\beta)}(\beta)=\max\{\psi^{i(\beta)}_{\alpha_j}(p)\mid p<c(\beta,j),\ j=1,2,\ldots,m(\beta)\}$. Let $u(\beta)=\max\{\varphi_\beta(y(\beta)),r^{i(\beta)}(\beta)\}$.

There are two cases:

Case 0a. $\omega(k(\beta))\leqslant r^{i(\beta)}(\beta)$, then

- ENUMERATE $\omega(k(\beta))$ in $A_{1-i(\beta)}$;
- Create a *link* (α_1,β);
- Initialise any $\gamma\not\leqslant\beta$ and go to stage $s+1$.

Case 0b. Otherwise, then

- Enumerate $y(\beta)$ into $A_{1-i(\beta)}$;
- ENUMERATE $\omega(k(\beta))$ in $A_{i(\beta)}$;
- Initialise any $\gamma\not\leqslant\beta$ and go to stage $s+1$.

β4. If $k(\beta)\neq-1$ and $y(\beta)=-1$, then define $y(\beta)$ to be a fresh odd number. We say that β receives attention at stage s, and for any γ, if $k(\beta)<k(\gamma)$, then reset γ, initialise any $\gamma\not\leqslant\beta$ and go to stage $s+1$.

β5. If $k(\beta)=-1$, then let $k(\beta)$ be a fresh number, initialise any $\gamma\not\leqslant\beta$, and go to stage $s+1$.

β6. Otherwise, let $\beta^\frown\langle3,\langle e(\beta),i(\beta)\rangle\rangle$ be eligible to act next.

Action phase of stage s.

Let β be the longest real strategy $\subset\xi$, let ξ^- be the longest node $\subset\xi$. Then β is a $\mathscr{P}$-strategy. Suppose that all real $\mathscr{M}$-strategies which are active at β are $\alpha_1,\alpha_2,\ldots,\alpha_m$ with $\alpha_1\subset\alpha_2\subset\ldots\subset\alpha_m$. By the assumption of ξ, $\xi=\xi^{-\frown}\langle(f_{\alpha_j},p_j)\rangle$ for some $j<m(\beta)$.

- For every $x\leqslant p_j$, define s_x to be the greatest stage $s'<s$ at which $\xi^\frown\langle(f_{\alpha_{j-1}},x)\rangle$ was visited or initialised, if s' exists, and define s_x to be the greatest stage $s'<s$ at which ξ was visited, otherwise;

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• Let p be the least x such that $\max\{\lambda_{\alpha_{j+1}}(x)[t]\neq\omega \mid s_x\leqslant t<s\}>\max\{\lambda_{\alpha_{j-1}}(x)[t]\neq\omega \mid t<s_x\}$;

• Let $\xi\hat{}\langle(f_{\alpha_{j+1}},p)\rangle$ be eligible to act next.

5 The Verification

In this section, we will prove that the construction satisfies all of the requirements. We first observe some properties which hold at the end of an arbitrarily given stage.

Proposition 5.1. (i) If there is a $\mathscr{P}$-strategy β such that $r^{i(\beta)}(\beta)[s-1]=-1\neq r^{i(\beta)}(\beta)[s]$, then $k(\beta)[s]\neq -1$, $r^{i(\beta)}(\beta)[s]\leqslant s$ and for any $k\geqslant k(\beta)[s]$, $\omega(k)[s]=\omega$;

(ii) If there is a $\mathscr{P}$-strategy β such that $r^{i(\beta)}(\beta)[s-1]=-1\neq r^{i(\beta)}(\beta)[s]$, then for any ξ, if $-1\neq k(\xi)[s-1]<k(\beta)[s-1](=k(\beta)[s])$, then $\omega(k(\beta))[s-1]>r^{i(\xi)}(\xi)[s]$;

(iii) At an odd stage s, if $\omega(k_s)\leqslant r^{i(\xi)}(\xi)$ for some $\mathscr{P}$-strategy ξ, then $k_s<k(\xi)$ and there is a unique β such that $\omega(k_s)\leqslant r^{i(\beta)}(\beta)$ and for any ξ, if $\xi\neq\beta$ and $\omega(k_s)\leqslant r^{i(\xi)}(\xi)$, then $k(\beta)<k(\xi)$;

(iv) For any real $\mathscr{M}$-strategies α and α', if $\alpha\subset\alpha'$ and α is active at α', then for any p and s. If $\lambda_{\alpha'}(p)[s]\neq\omega$, then $\lambda_{\alpha}(p)[s]<\lambda_{\alpha'}(p)[s]$;

(v) For a real $\mathscr{P}$-strategy β with $m(\beta)\neq 0$, if $j<m(\beta)$ and $c(\beta,j)[s]\downarrow$, then $c(\beta,j+1)[s]\downarrow<c(\beta,j)[s]$;

(vi) If a real $\mathscr{P}$-strategy β is reset at stage s, then either (a) or (b) below holds:

(a) There is a β' such that $-1\neq k(\beta')[s]<k(\beta)[s]$ and $\omega(k_s)\leqslant r^{i(\beta')}(\beta')[s]$;

(b) There is a β' such that $-1\neq k(\beta')[s]<k(\beta)[s]$ and $y(\beta')[s-1]=-1\neq y(\beta')[s]$.

Proof. These properties are immediate from the construction, and we leave them to the readers. □

Secondly, we investigate some properties which hold at the end of the construction.

Proposition 5.2. Given a real $\mathscr{P}$ strategy β,

(i) β is initialised only finitely many times if and only if $\lim_s k(\beta)[s]\downarrow=k(\beta)<\omega$;

(ii) If $\lim_s k(\beta)[s]\downarrow=k(\beta)<\omega$, then $\lim_s y(\beta)[s]\downarrow<\omega$, $\lim_s r^{i(\beta)}(\beta)[s]\downarrow=r^{i(\beta)}(\beta)<\omega$;

(iii) If $\lim_s k(\beta)[s]\downarrow=k(\beta)<\omega$, then β is reset only finitely many times and β receives attention only finitely many times.

Proof. (i) is immediate from the construction, and (iii) follows from (ii). It suffices to prove (ii). If β is visited only finitely many times, then clearly (ii) holds. Thus we suppose that β will be visited infinitely many times from now on.

Choose s_0 to be the least stage such that for any $s\geqslant s_0$, $k(\beta)[s]=k(\beta)[s_0]$. By the definition of the thresholds, there are only finitely many β's such that $-1\neq k(\beta')[s_0]<k(\beta)[s_0]$.

For a β' with $-1\neq k(\beta')[s_0]<k(\beta)[s_0]$, if $\lim_s k(\beta')[s]\neq k(\beta')[s_0]$, then let $s(\beta')$ be the least stage $>s_0$ at which $k(\beta')\neq k(\beta')[s_0]$, then β' will never reset β at a stage $>s(\beta')$. Let $s_1=\max\{s_0,s(\beta') \mid -1\neq k(\beta')[s_0]<k(\beta)[s_0]\ \&\ \lim_s k(\beta')[s]\neq k(\beta')[s_0]\}$.

Suppose by induction for a β' with $-1\neq k(\beta')[s_0]<k(\beta)[s_0]$ and $\lim_s k(\beta')[s]\downarrow=k(\beta')[s_0]$ that

(a) $\lim_s y(\beta')[s]\downarrow=y(\beta')<\omega$;

(b) $\lim_s r^{i(\beta')}(\beta')[s]=r^{i(\beta')}(\beta')<\omega$.

Let $t(\beta')$ be the least stage $>s_1$ such that for any $s\geqslant t(\beta')$, both $y(\beta')[s]=y(\beta')[t(\beta')]$ and $r^{i(\beta')}(\beta')[s]=r^{i(\beta')}(\beta')[t(\beta')]$ hold. Then by (vi) of proposition 5.1, β' will never receive attention at a stage $s\geqslant t(\beta')$.

Let $s_2=\max\{s_1,t(\beta') \mid -1\neq k(\beta')[s_0]<k(\beta)[s_0]\ \&\ \lim_s k(\beta')[s]\downarrow=k(\beta')[s_0]\}$. Then β will never be reset at a stage $>s_2$. Therefore, $\lim_s y(\beta)[s]\downarrow=y(\beta)(\neq -1)<\omega$.

Let s_3 be the least stage such that for any $s\geqslant s_3$, $y(\beta)[s]=y(\beta)[s_3]$. Then by the choice of s_3, $y(\beta)[s_3]\notin A_{1-i(\beta)}[s_3]$.

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If program 0 of β 3 will never occur at a stage $>s_3$, then for any $s \geqslant s_3$, $r^{i(\beta)}(\beta)[s]=-1$.

Suppose that s_4 is the least stage $>s_3$ at which program 0 of β 3 occurs. Then β defines its $A_{i(\beta)}$-restraint $r^{i(\beta)}(\beta)[s_4]$. We now claim that, for any $s \geqslant s_4$, $r^{i(\beta)}(\beta)[s]=r^{i(\beta)}(\beta)[s_4]$.

By the choice of s_3 and s_4, $r^{i(\beta)}(\beta)[s_4]$ will never be injured at odd stages $>s_4$. By the assumption of stage s_4, there is no $\xi \subset \beta$ which will enumerate an $x \leqslant u(\beta)[s_4]$ into A at a stage $>s_4$. And by the initialisation at stage s_4, any γ with $\gamma \nleqslant \beta$ is initialised, and then if some γ with $\gamma \nleqslant \beta$ enumerates an x into either $A_{i(\beta)}$ or A, then $x>s_4$. And by the choice of s_0, any $\gamma <_L \beta$ will never be visited at a stage $>s_4$. Therefore, for any $s \geqslant s_4$, $r^{i(\beta)}(\beta)[s]=r^{i(\beta)}(\beta)[s_4]$. Now (ii) follows.

Proposition 5.3. (i) For any k, $\lim_s \omega(k)[s]\downarrow=\omega(k)<\omega$; (ii) $K \leqslant_T A_0 \oplus A_1$.

Proof. (ii) follows from (i) and the ω-rules.

For (i), suppose to the contrary that n is the least k such that $\lim_s \omega(k)[s]=\infty$. By the construction and by the ω-rules, the unique possibility is that $\omega(n)$ is enumerated infinitely many times by some $\mathscr{P}$-strategies. By the choice of n and by the definition of the thresholds, there is a fixed $\mathscr{P}$-strategy β such that $\lim_s k(\beta)[s]\downarrow=n$, and β enumerates $\omega(n)$ infinitely many times. By β3 of program β, β enumerates $\omega(n)$ in A_i for some $i=0$, 1 only if β imposes a new $A_{i(\beta)}$-restraint. By proposition 5.2 (ii), $\lim_s r^{i(\beta)}(\beta)[s]\downarrow=r^{i(\beta)}(\beta)<\omega$. Thus for any k, $\lim_s \omega(k)[s]\downarrow=\omega(k)<\omega$. The proposition follows. □

By the full construction, the priority tree T is a c.e. set of all strategies which appear in the full construction. We define *the true path* TP of the construction to be the subset of all $\alpha \in T$ such that α is visited at infinitely many stages and there are only finitely many stages at which some $\beta <_L \alpha$ is visited.

Proposition 5.4 (Finite Initialisation Proposition). Given a node $\xi \in TP$,

(i) ξ is initialised only finitely many times;

(ii) ξ is visited at infinitely many stages.

Proof. Clearly the proposition holds for the root node. Suppose by induction that the proposition holds for every $\eta \subset \xi$. By proposition 5.2, choose s_0 to be the least stage such that, for any $\eta \subset \xi$, η will never be initialised or reset, or will never receive attention at a stage $\geqslant s_0$. Let ξ^- be the longest $\eta \subset \xi$. There are four cases:

Case A. ξ^- is a real $\mathscr{S}$-strategy.

This case is immediate from the construction.

Case B. ξ^- is a real $\mathscr{M}$-strategy.

Clearly (i) holds. Suppose to the contrary that (ii) fails to hold. Then let s_1 be the least stage $>s_0$ such that, for any $s \geqslant s_1$, if ξ is visited at stage s, then a link (ξ^-, β) is travelled for some real $\mathscr{P}$-strategy β at stage s, by α2 of the construction, this is impossible. Hence (ii) holds for ξ.

Case C. $\xi^-=\beta$ is a real $\mathscr{P}$-strategy.

If $m(\beta)=0$, then clearly the proposition holds for ξ. Assume $m(\beta)\neq 0$. By proposition 5.2, $\lim_s y(\beta)[s]\downarrow=y(\beta)<\omega$ and $\lim_s r^{i(\beta)}(\beta)[s]\downarrow=r^{i(\beta)}(\beta)<\omega$. Let s_1 be the least stage $>s_0$ such that, for any $s \geqslant s_1$, $y(\beta)[s]=y(\beta)[s_1]$, $r^{i(\beta)}(\beta)[s]=r^{i(\beta)}(\beta)[s_1]$ and β will never receive attention at stage s. Therefore by the construction, for any $\gamma \supset \beta$, γ will never be initialised by strategies $\subseteq \beta$ at a stage $>s_1$.

Now by the definition of TP, both (i) and (ii) hold for ξ.

Case D. $\xi^-=\gamma$ is a virtual strategy.

Let β be the longest real strategy $\subset \gamma$. Then by the definition of the priority tree T, β is a $\mathscr{P}$-strategy and $m=m(\beta)\neq 0$, and $\beta \subset \xi^- \subset \xi$.

If $\xi^- \neq \beta^\frown\langle(2,-j)\rangle$ for some $j \in \{1,2,\ldots,m\}$, then by the assumption of $\xi \in TP$ and by the λ-rules, both (i) and (ii) hold for ξ.

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Otherwise, $\xi=\beta^\frown\langle(2,-j)\rangle^\frown\langle(f_{\alpha_j},p)\rangle$ for some p. By the construction and by the assumption of $\xi\in TP$; both (i) and (ii) hold for ξ.

Thus in any case, the proposition holds for ξ. □

Proposition 5.5 (The Existence of the TP Proposition). Given a node $\xi\in TP$, there is a such that $\xi^\frown\langle a\rangle\in TP$.

Proof. The unique nontrivial case is that $\xi=\beta^\frown\langle(2,-j)\rangle$ for some $\mathscr{P}$-strategy β and for some $j\leqslant m(\beta)$. □

By proposition 5.4, choose s_0 to be minimal such that ξ will never be initialised at a stage $\geqslant s_0$. By the assumption of ξ, $\lim_s\varphi_\beta(y(\beta))[s]\downarrow=\varphi_\beta(y(\beta))<\omega$, and then $\lim_s c(\beta,m)[s]\downarrow=c(\beta,m)<\omega$ for $m=m(\beta)$.

Suppose that j is the least i such that $c(\beta,i)[s]\downarrow<\omega$. By the construction, for any $i\in\{j+1,\ldots,m(\beta)\}$, Step ic of program $-i$ of $\beta 3$ occurs only finitely many times, and Step jc of program $-j$ of $\beta 3$ occurs at infinitely many stages. Then by the construction, $\lambda_{\alpha_j}(c(\beta,j))$ will be unbounded. And then there is a $p\leqslant c(\beta,j)$ such that $\beta^\frown\langle(2,-j)\rangle^\frown\langle(f_{\alpha_j},p)\rangle$ will be on the true path TP. The proposition follows. □

Proposition 5.6 (Finite Injury Along TP Proposition). For any requirement $\mathscr{X}\neq\mathscr{R}$, there exists a strategy α such that $\alpha\in TP$, and either $\mathscr{X}$ is active at ξ for any ξ with $\alpha\subset\xi\in TP$ or $\mathscr{X}$ has been satisfied at ξ for any ξ with $\alpha\subset\xi\in TP$.

Proof. For $\mathscr{M}_0$, let λ be the root node. If $\alpha=\lambda^\frown\langle(1,0)\rangle\in TP$, then for any ξ, if $\alpha\subset\xi\in TP$, then $\mathscr{M}_0$ has been satisfied at ξ. If there is a $\mathscr{P}$-strategy β such that $\alpha\subseteq\xi_0=\beta^\frown\langle(2,-j)\rangle^\frown\langle(f_\lambda,p)\rangle\in TP$ for some j and for some p, then for any ξ, if $\xi_0\subset\xi\in TP$, then $\mathscr{M}_0$ has been satisfied at ξ. Otherwise, for any ξ, if $\alpha\subset\xi\in TP$, then $\mathscr{M}_0$ is active at ξ.

Suppose by induction that α is the shortest real strategy $\in TP$ such that for every requirement $\mathscr{X}'<\mathscr{X}$ and $\mathscr{X}'\neq\mathscr{R}$, then either $\mathscr{X}'$ has been satisfied at ξ for any ξ with $\alpha\subseteq\xi\in TP$ or $\mathscr{X}'$ is active at ξ for any ξ with $\alpha\subseteq\xi\in TP$.

Choose ξ_0 to be the longest real strategy $\subset\alpha$ which works on $\mathscr{X}$. If $\mathscr{X}=\mathscr{I}_e$ for some e, let $\xi_1=\xi_0{}^\frown\langle a\rangle\in TP$, then for any ξ, if $\xi_1\subset\xi\in TP$, then $\mathscr{X}$ has been satisfied at ξ.

If $\mathscr{X}=\mathscr{P}_e^i$ for some e,i, then by the maximality of ξ_0, $\xi_0{}^\frown\langle a\rangle\in TP$ for some $a\in\{(0,\langle e,i\rangle),(1,\langle e,i\rangle),(3,\langle e,i\rangle)\}$. Thus in any case, $\mathscr{X}$ has been satisfied at ξ for any ξ with $\xi_1\subset\xi\in TP$.

If $\mathscr{X}=\mathscr{M}_e$ for some e, then by the proof of case $\mathscr{M}_0$. The proposition follows.

Proposition 5.7 (Possible Outcomes Along TP Proposition). (i) For a real $\mathscr{I}$-strategy δ, if $\delta^\frown\langle(0,e(\delta))\rangle\in TP$, then $J^\delta=\omega^\delta$; (ii) If $\xi^\frown\langle(f_\alpha,p)\rangle\in TP$ for some ξ, α and p, then $\lambda_\alpha(p)[s]$ will be unbounded; (iii) For a real $\mathscr{P}$-strategy β, if $\beta^\frown\langle(1,e)\rangle\in TP$ for $e=\langle e(\beta),i(\beta)\rangle$, then $\lim_s\varphi_\beta(y(\beta))[s]=\infty$; (iv) For a real $\mathscr{M}$-strategy α, if $\alpha^\frown\langle(0,e(\alpha))\rangle\in TP$, then $\Lambda_\alpha(A)$ will be built infinitely often.

Proof. This proposition is immediate from the construction. □

Proposition 5.8 ($\mathscr{I}$ Satisfaction Proposition). For every e, $\mathscr{I}_e$ will be satisfied.

Proof. Let $\delta\in TP$ be the real $\mathscr{I}_e$-strategy. By proposition 5.4, $\lim_s b(\delta)[s]\downarrow=b(\delta)<\omega$. By proposition 5.7, $J^\delta=J^{[2e]}$. By program δ of the construction, every $x>b(\delta)$ and $x\in J^\delta$ will be enumerated into A eventually. Then $J^{[2e]}\subseteq^* A$. And by the construction, $A^{[2e]}=^* J^{[2e]}$. Hence $\mathscr{I}_e$ will be satisfied. □

Proposition 5.9 ($\mathscr{M}$ Satisfaction Proposition). For every e, if $\Psi_e^0(A_0,A)=\Psi_e^1(A_1,A)=g_e$ is total, then $g_e\leqslant_T A$.

Proof. By proposition 5.6, let ξ_0 be the shortest node $\in TP$ such that either $\mathscr{M}_e$ has been satisfied at ξ for any ξ with $\xi_0\subset\xi\in TP$ or $\mathscr{M}_e$ is active at ξ for any ξ with $\xi_0\subset\xi\in TP$. Let α be the longest real $\mathscr{M}_e$ strategy $\subset\xi_0$. By the assumption of the proposition, $\alpha^\frown\langle(0,e(\alpha))\rangle\subseteq\xi_0$. By the choice of ξ_0 and by proposition 5.7, $\Lambda_\alpha(A)$ will be built infinitely often. By the $\mathscr{M}$-strategy α, if $\Lambda_\alpha(A)$ is a total function, then $\Lambda_\alpha(A)=g_e$. $\mathscr{M}_e$

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will be satisfied. □

Suppose to the contrary that p is the least number such that $\lambda_\alpha(p)$ will be unbounded during the full construction. By the assumption of the proposition, $\lambda_\alpha(p)$ is enumerated by some $\beta\subseteq\alpha$ only finitely many times. Therefore there is a $\mathscr{P}$-strategy β such that $\alpha\hat{}\langle(0,e(\alpha))\rangle\subseteq\beta\subset\beta\hat{}\langle(2,-j)\rangle\hat{}\langle(f_\alpha,p)\rangle\subseteq\xi_0$ for some j. Therefore program $-j$ of $\beta 3$ occurs infinitely many times. By the choice of ξ_0, $\lim_s c(\beta,j)[s]\downarrow=c(\beta,j)<\omega$. By the assumption that $\Psi_e^0(A_0,A)=\Psi_e^1(A_1,A)=g_e$ is total, Step jc of program $-j$ of $\beta 3$ occurs only finitely many times. Therefore $\beta\hat{}\langle(2,-j)\rangle<_L TP$. A contradiction.

Thus $A_\alpha(A)$ is a total function. $\mathscr{M}_e$ is satisfied.

Proposition 5.10 (**$\mathscr{P}$ Satisfaction Proposition**). For every e, i, $\mathscr{P}_e^i$ is satisfied.

Proof. By proposition 5.7, choose β to be the longest real $\mathscr{P}_e^i$-strategy $\in TP$. Then $\beta\hat{}\langle a\rangle\in TP$ for $a\in\{(k,\langle e,i\rangle)\mid k=0,1,3\}$. By proposition 5.4, $\lim_s k(\beta)[s]=k(\beta)<\omega$, and then by proposition 5.2, $\lim_s y(\beta)[s]\downarrow=y(\beta)<\omega$. Let s_0 be the least stage such that for any $s\geqslant s_0$, $k(\beta)[s]=k(\beta)[s_0]$ and choose s_1 to be the least stage such that for any $s\geqslant s_1$, $y(\beta)[s]=y(\beta)[s_1]$. Then by the construction $s_0<s_1$. There are three cases:

Case 1. $\beta\hat{}\langle(0,\langle e,i\rangle)\rangle\in TP$.

Clearly $y(\beta)\in A_{1-i(\beta)}$. Let s_2 be the stage at which we enumerate $y(\beta)$ into $A_{1-i(\beta)}$. Then $s_1<s_2$.

If $m(\beta)=0$, then $\Phi_\beta(A;y(\beta))[s_2]\downarrow=0\neq1=A_{1-i(\beta)}(y(\beta))$ and by the assumption of β at stage s_2, there is no $\xi\subset\beta$ which will enumerate an $x\leqslant\varphi_\beta(y(\beta))[s_2]$ into A at a stage $>s_2$, and by the initialisation at stage s_2, there is no any strategy $\gamma\not\leqslant\beta$ which will enumerate an $x\leqslant s_2$ into A at a stage $>s_2$. Therefore $\Phi_\beta(A;y(\beta))[s_2]\downarrow=0$ will be preserved forever. Thus $\Phi_\beta(A;y(\beta))\downarrow=0\neq1=A_{1-i(\beta)}(y(\beta))$ will be preserved forever. $\mathscr{P}_e^i$ is satisfied.

Assume $m(\beta)=m>0$.

Suppose that we travel a link (α,β) for some real $\mathscr{M}$-strategy α which is active at β at stage s_2. Suppose that the current link (α,β) was created at stage s_3, then $s_1<s_3<s_2$. And by the construction, $\Phi_\beta(A;y(\beta))[s_3]\downarrow=0$. By the choice of stages s_3 and s_1, $r^{i(\beta)}(\beta)[s_3]$ will never be injured at a stage $>s_3$.

Suppose that all real $\mathscr{M}$-strategies which are active at β are $\alpha_1,\alpha_2,\ldots,\alpha_m$ with $\alpha_1\subset\alpha_2\subset\ldots\subset\alpha_m$. Then for every $j\in\{1,2,\ldots,m\}$, $c(\beta,j)[s_3]\downarrow$, and for any $p<c(\beta,j)[s_3]$, $A_{\alpha_j}(A;p)\downarrow=\Psi_{\alpha_j}^{i(\beta)}(A_{i(\beta)},A;p)[s_3]$ is β-believable, and for any $p\geqslant c(\beta,j)[s_3]$, $\lambda_{\alpha_j}(p)[s_3]=\omega$. By the construction at stage s_3 and by the $A_{i(\beta)}$-restraint $r^{i(\beta)}(\beta)[s_3]$, for every $j\in\{1,2,\ldots,m\}$, and for any $p<c(\beta,j)[s_3]$, $A_{\alpha_j}(A;p)[s_3]\downarrow=\Psi_{\alpha_j}^{i(\beta)}(A_{i(\beta)},A;p)[s_3]$ holds forever. Therefore $\Phi_\beta(A;y(\beta))[s_3]\downarrow=0$ will never be injured by the building of $A_{\alpha_j}(A)$ for any $j\in\{1,2,\ldots,m\}$, at a stage $>s_3$, and by the initialisation of stage s_3, $\Phi_\beta(A;y(\beta))[s_3]\downarrow=0$ will be preserved forever. And then $\Phi_\beta(A;y(\beta))\downarrow=0\neq1=A_{1-i(\beta)}(y(\beta))$ will be preserved forever. $\mathscr{P}_e^i$ is satisfied.

If we enumerate $y(\beta)$ into $A_{1-i(\beta)}$ at stage s_2, then by the same argument as above, $\Phi_\beta(A;y(\beta))[s_2]\downarrow=0$ and it will be preserved forever, and then $\Phi_\beta(A;y(\beta))[s_2]\downarrow=0\neq1=A_{1-i(\beta)}(y(\beta))$. $\mathscr{P}_e^i$ is satisfied.

Case 2. $\beta\hat{}\langle(1,\langle e,i\rangle)\rangle\in TP$.

By proposition 5.8, $\varphi_\beta(y(\beta))$ will be unbounded. $\mathscr{P}_e^i$ is satisfied.

Case 3. $\beta\hat{}\langle(3,\langle e,i\rangle)\rangle\in TP$.

By proposition 5.8, either $\varphi_\beta(y(\beta))$ will be unbounded or $\Phi_\beta(A;y(\beta))\neq0=A_{1-i(\beta)}(y(\beta))$. In either case, $\mathscr{P}_e^i$ is satisfied.

Therefore, in any case, $\mathscr{P}_e^i$ is satisfied. □

This completes the proof of the theorem. □

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一个高的钻石定理

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摘要　证明存在一个保持最大元 1 的可计算枚举高度的钻石格.

关键词　可计算性理论,可计算枚举度,可计算枚举度,Turing 归约(图灵归约),相对可计算性.

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