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# Functional Compositions via Shifting Operators for Bézier Patches and Their Applications*

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**Abstract** There are two kinds of Bézier patches which are represented by different base functions, namely the triangular Bézier patch and the rectangular Bézier patch. In this paper, two results about these patches are obtained by employing functional compositions via shifting operators. One is the composition of a rectangular Bézier patch with a triangular Bézier function of degree 1, the other is the composition of a triangular Bézier patch with a rectangular Bézier function of degree 1×1. The control points of the resultant patch in either case are the linear convex combinations of the control points of the original patch. With the shifting operators, the respective procedure becomes concise and intuitive. The potential applications of the two results include conversions between two kinds of Bézier patches, exact representation of a trimmed surface, natural extension of original patches, etc.

**Key words** Rectangular Bézier patch, triangular Bézier patch, functional composition, computer aided geometric design, de, Casteljau algorithm.

Both rectangular and triangular Bézier patches are widely used in the field of computer-aided geometric design. The two kinds of patches, however, adopt different base functions and conform to different topological structures[1]. It is very interesting to exploit the intrinsic relationship between them. This research will be helpful for solved the problems such as conversions between two kinds of patches[2~5], exact trimmed surface, natural extension of Bézier patch, etc.

Functional compositions of polynomial in terms of the Bernstein function form were studied by DeRose *et al.*[6,7] In particular, Ref. [6] concerns the Bézier simplex composition and demonstrates some practical algorithms. Later, compositions between Bézier simplex and simploid were also studied by the same group, employing blossoming method[7]. Since they dealt with the problem of composition under general situation, the proposed algorithms are not easy either to be understood or to be implemented. In this paper, functional compositions for two kinds of Bézier patches are studied using shifting operators as tools[8]. With the shifting operators the respective procedure becomes concise and intuitive. The resultant algorithm is also easy to encode.

The rest of the paper is organized as follows. In Section 1, the notations of Bézier patch representations via shifting operators are given. The generalized de'Casteljau algorithm is also introduced through a 1D example.

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Then the details of functional composition are presented in Section 2. Some application examples are given in the followed section and conclusions are drawn in the last section.

## 1 Preliminaries

Bernstein polynomial representation via shifting operator was first introduced by Chang[8]. Let $\mathbf{R}(u,v,w)$ be a triangular Bézier patch of degree $n$. Its polynomial representation is

$$\mathbf{R}(u,v,w)=\sum_{i+j+k=n}\mathbf{R}_{ijk}B_{ijk}^{n}(u,v,w), \tag{1}$$

where $B_{ijk}^{n}=\binom{n}{i,j,k}u^{i}v^{j}w^{k}$ and $i,j,k\geqslant 0$; $u+v+w=1$ and $u,v,w\geqslant 0$. It is well known that its parametric domain is a planar triangular region. Let $E_u$, $E_v$, $E_w$ be noted as shifting operators whose definitions are

$$\mathbf{E}_u\mathbf{R}_{ijk}=\mathbf{R}_{i+1,j,k},\quad E_v\mathbf{R}_{ijk}=\mathbf{R}_{i,j+1,k},\quad E_w\mathbf{R}_{ijk}=\mathbf{R}_{i,j,k+1}. \tag{2}$$

With the above notations, the triangular Bézier patch $\mathbf{R}(u,v,w)$ can be rewritten as

$$\mathbf{R}(u,v,w)=(uE_u+vE_v+wE_w)^n\mathbf{R}_{000}. \tag{3}$$

Similarly, a rectangular Bézier patch $\mathbf{P}(s,t)$ can be represented as

$$\mathbf{P}(s,t)=[(1-s)I+sE_s]^m[(1-t)I+tE_t]^n\mathbf{P}_{00}, \tag{4}$$

where $(s,t)\in[0,1]^2$ and the shifting operators $I$, $E_s$, $E_t$ are defined as

$$I\mathbf{P}_{ij}=\mathbf{P}_{ij},E_s\mathbf{P}_{ij}=\mathbf{P}_{i+1,j},E_t\mathbf{P}_{ij}=\mathbf{P}_{i,j+1}. \tag{5}$$

The correctness of Equations (3) and (4) can be proved easily by binomial and trinomial expansion laws.

The generalized de'Casteljau algorithm is designed to compute the Bézier control points of a sub-curve which is a portion of the original Bézier curve[1]. This algorithm can be derived more intuitively through shifting operators. Let

$$\mathbf{P}(u)=[(1-u)I+uE_u]^n\mathbf{P}_0 \tag{6}$$

be a Bézier curve, which is defined on interval $[0,1]$. Suppose $[u_0,u_1]\in[0,1]$, and

$$u(t)=(1-t)u_0+tu_1 \qquad t\in[0,1],$$

then the composition of $\mathbf{P}(u)$ and $u(t)$ is

$$\begin{aligned}\mathbf{P}(t)&=\mathbf{P}(u(t))\\&=[(1-u(t))I+u(t)E_u]^n\mathbf{P}_0\\&=[\mathbf{A}_0(1-\mathbf{t})+\mathbf{A}_1\mathbf{t}]^n\,\mathrm{P}_0\\&=\sum_{k=0}^{n}\mathrm{P}B_{k,n}(t)\end{aligned}$$

where $A_i=(1-u_i)I+u_iE_u, i=0,1$; $\mathbf{P}_k=A_0^{n-k}A_1^k\mathbf{P}_0$. It is the generalized de'Casteljau algorithm by means of shifting operators.

## 2 Composition of Bézier Patch with Linear Function

### 2.1 Composition of a rectangular Bézier patch with triangular linear function

Let $A_0A_1A_2$ be a triangular region defined in the parametric domain $[0,1]^2$ of a rectangular Bézier patch $\mathbf{P}(s,t)$ which is expressed in the form of Equation (4) (Fig. 1). The region $A_0A_1A_2$ can be parameterized with barycentric coordinates:

$$s(u,v,w)=s_0u+s_1v+s_2w \tag{7}$$

$$t(u,v,w)=t_0u+t_1v+t_2w \tag{8}$$

Where $u+v+w=1$ and $u,v,w\geqslant 0$. It is easy to show that rectangular Bézier patch $\mathbf{P}(\mathbf{s},\mathbf{t})$ defined on $A_0A_1A_2$ can be represented as a triangular Bézier patch, whose control points are determined as follows:

$$\begin{aligned}\mathbf{P}(u,v,w)&=\mathbf{P}(s(u,v,w),t(u,v,w))\\&=[(1-s(u,v,w))I+s(u,v,w)E_s]^m[(1-t(u,v,w))I+t(u,v,w)E_t]^n\mathbf{P}_{00}\\&=(As_0u+As_1v+As_2w)^m(At_0u+At_1v+At_2w)^n\mathbf{P}_{00}\\&=\sum_{i+j+k=m+n}\mathbf{P}_{ijk}B_{ijk}^{m+n}(u,v,w)\end{aligned}$$

Where $As_i=(1-s_i)I+s_iE_s$, $At_j=(1-t_j)I+t_jE_t$, $i=0,1,2$ and

$$\mathbf{P}_{ijk}=\frac{\sum\binom{m}{i_s,j_s,k_s}\binom{n}{i_t,j_t,k_t}As_0^{i_s}As_1^{j_s}As_2^{k_s}A^{i_t}t_0At_1^{j_t}A^{k_t}t_2\mathbf{P}_{00}}{\binom{m+n}{i,j,k}}.$$

The "$\sum$" is subject to $i_s+j_s+k_s=m$, $i_t+j_t+k_t=n$, $i_s+i_t=i$, $j_s+j_t=j$, $k_s+k_t=k$. We then compute the control points $\mathbf{P}_{ijk}$ forwardly through generalized de'Casteljau algorithm.

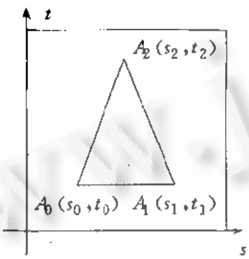


Fig. 1. Triangular Bézier function of degree 1 in $[0,1]^2$

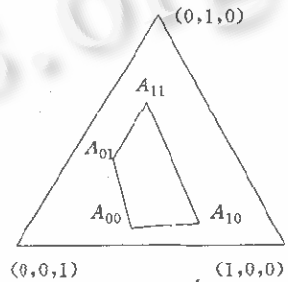


Fig. 2. Rectangular Bézier function of degree $1\times1$ in *uvw* plane

### 2.2 Composition of a triangular Bézier patch with bilinear functions defined on a quadrilateral region

Let $\mathbf{A}_{00}\mathbf{A}_{10}\mathbf{A}_{01}\mathbf{A}_{11}$ be a quadrilateral region defined in the parametric domain of the triangular Bézier patch $\mathbf{R}(u,v,w)$ as Equation (3) (See Fig. 2). Note $A_{ij}=(u_{ij},v_{ij},w_{ij})$, $i,j=0,1$. The quadrilateral region can be parametrized by using the following bilinear function:

$$\mathbf{A}(s,t)=[(1-s)I-sE_s][(1-t)I+tE_t]\mathbf{A}_{00}.$$

After substituting Equation (9) into (3), we can get a rectangular Bézier patch with degree of $n\times n$ as follows:

$$\begin{aligned}\tilde{\mathbf{G}}(s,t)&=\mathbf{R}(u(s,t),v(s,t),w(s,t))\\&=[(1-s)(1-t)A_{00}+(1-s)tA_{01}+s(1-t)A_{10}+stA_{11}]^n\mathbf{R}_{000}\\&=\sum_{p=0}^{n}\sum_{q=0}^{n}\tilde{\mathbf{G}}_{pq}B_{p,n}(s)B_{q,n}(t)\end{aligned}$$

where $A_{ij}=(u_{ij}E_u+v_{ij}E_v+w_{ij}E_w)$, $i,j=0,1$; and

$$\tilde{\mathbf{G}}_{pq}=\frac{\sum\binom{n}{i,j,k,l}A_{00}^{i}A_{01}^{j}A_{10}^{k}A_{11}^{l}\mathbf{R}_{000}}{\binom{n}{p,n-p}\binom{n}{q,n-q}}.$$

The "$\sum$" is subject to $i-j+k+l=n$, $k+l=p$, $j+l=q$.

## 3 Some Applications of the Proposed Algorithms

### 3.1 Conversions between triangular and rectangular Bézier patches

By employing the proposed algorithms, it is easy to convert a triangular Bézier patch into three non-degenerate rectangular Bézier patches. This is achieved by splitting the triangular domain into three rectangular ones as shown in Fig. 3. On the other hand, we can also convert a rectangular Bézier patch into two triangular Bézier patches. In this case, the square domain needs to be subdivided into two triangular ones as shown in Fig. 4.

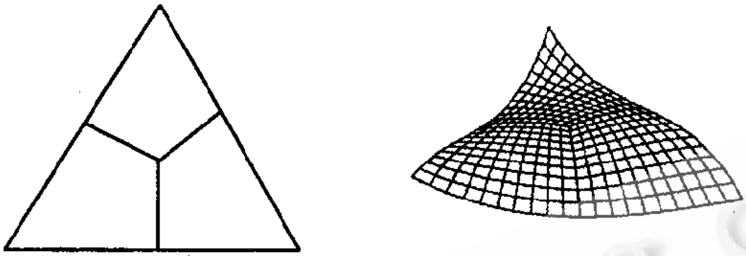

Fig. 3.　Converting a triangular Bézier patch into three triangular patches

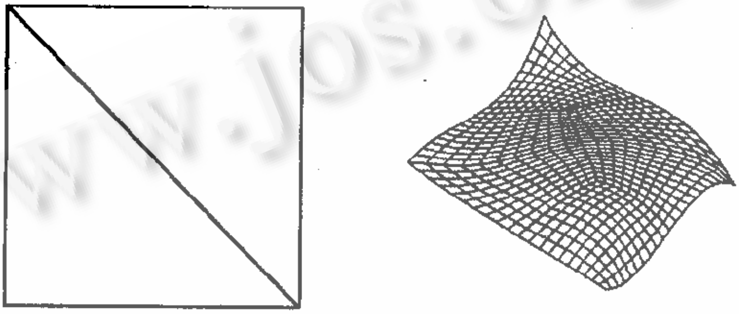

Fig. 4.　Converting a rectangular Bézier patch into two triangular patches

### 3.2 Exact representation of a trimmed Bézier patch

With the above algorithms, we can obtain a precise representation of a trimmed Bézier surface patch. The key is to find the planar polygonal region in the parametric domain of a Bézier patch over which the trimmed surface is defined. Figure 5 shows the result of trimming a rectangular patch from a triangular patch of degree 3, where the triangular patch is drawn as dash lines, the trimmed patch is drawn as solid lines. Figure 6 shows the result of trimming an $n$-sides patch from a rectangular Bézier patch . The trimmed patch is a combination of $n$ triangular Bézier patches.

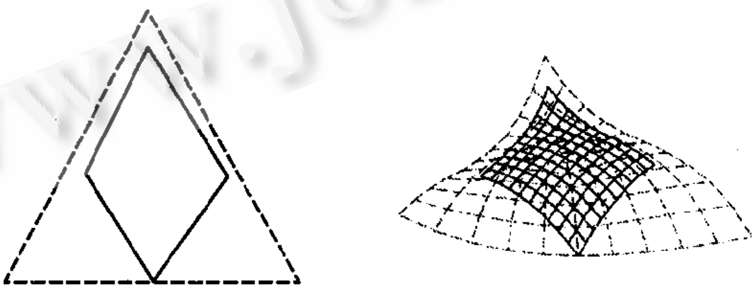

Fig. 5.　Trimming a rectangular patch from a triangular one

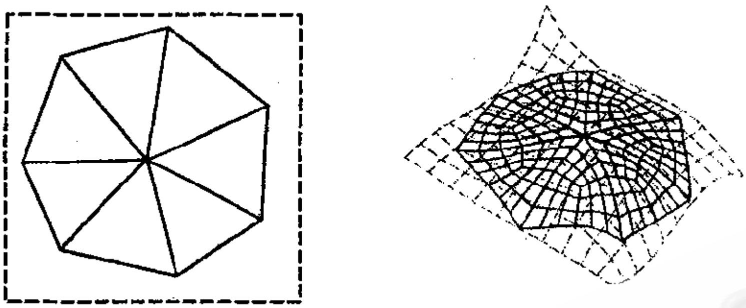

Fig. 6. Trimming 7 triangular patches from a rectangular one

### 3.3 Extended Bézier patch

When a composite patch is defined outside of the parametric domain over which the original surface is defined, the proposed algorithm can still work properly though on occasions it may be numerically unstable. The resultant patch can be regarded as a natural extension of the original patch with $C^{\infty}$ continuity. Here are three examples. Figure 7 shows an example, which is an extension of a rectangular patch along its boundary curves and corner points. The original patch is displayed as dash lines, the extension part is drawn in solid lines. In Figure 8, the original patch is a rectangular patch, and the four attached patches are four triangular ones. Figure 9 shows another example, where the original patch is a triangular patch, while the three attached patches are rectangular ones.

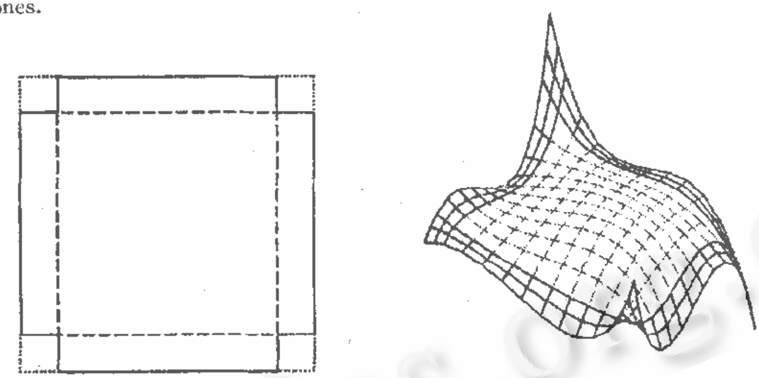

Fig. 7. Extension of a rectangular Bézier patch along its boundary curves and corner points

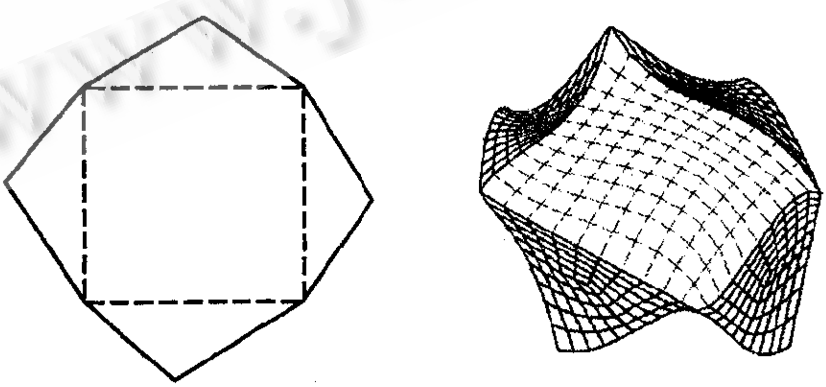

Fig. 8. Extension of a rectangular Bézier patch by attaching four triangular ones

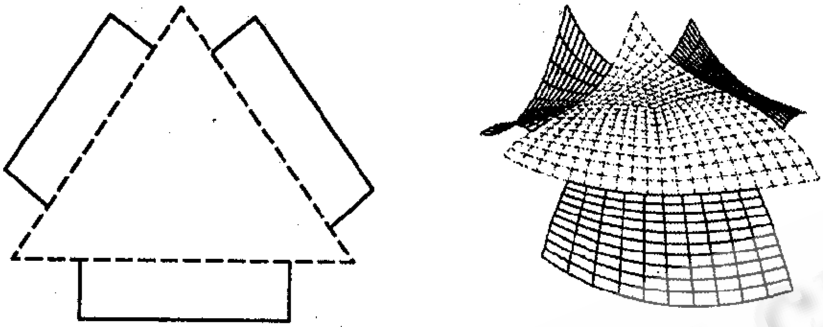

Fig. 9.　Extension of a triangular Bézier patch by attaching three rectangular ones

## 4　Conclusions

In this paper, the relations between rectangular and triangular Bézier surface patches are well studied by means of shifting operators and functional composition. The deduced results are intuitive and can be encoded easily. Some applications are demonstrated including conversions between rectangular and triangular Bézier patches, exact representation of trimmed patches, natural extension of Bézier surface patches with $C^{\infty}$ continuity. As a future research topic, we will exploit the applications of generalized de'Casteljau algorithm in the fields of CAD and computer graphics.

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# Bézier曲面的函数复合及其应用

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**摘要** 目前有两种常用的Bézier曲面片，分别称为三角和四边Bézier曲面片，它们分别用不同的基函数表示. 本文通过移位算子和函数复合的方法，得到了两个关于这两种Bézier曲面片的结果. 一个是四边Bézier曲面片与一次三角Bézier函数的复合，另一个是三角Bézier曲面片与双线性四边Bézier函数的复合. 在每一种情况中，复合所得到的Bézier曲面片的控制顶点是原来Bézier曲面片的控制顶点的线性组合. 移位算子的应用使得相应的推导过程变得简洁和直观. 这两个结果的应用包括：两种Bézier曲面片间的转化、裁剪Bézier曲面片的精确表示、Bézier曲面片的自然延拓等.

**关键词** 四边Bézier曲面片，三边Bézier曲面片，函数复合，计算机辅助几何设计，de'Casteljau算法.

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