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P3P问题的多解现象的概率研究*

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A Probabilistic Study on the Multiple Solutions of the P3P Problem

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Abstract: Under some special conditions, the P3P problem can have 1, 2, 3 and 4 solutions, and if the 3 control points and the optical center lie on a circle, the problem is indeterminate. In this paper, by the Monte Carlo approach of up to 1 million samples, it is shown that the probabilities of the P3P problem with one solution, two solutions, three solutions, and four solutions are respectively 0.9993, 0.0007, 0.0000, 0.0000. The result confirms the well-known fact that in the most cases, the P3P has a unique solution.
Key words: P3P problem; multiple solutions; Monte Carlo approach

摘 要: 一般情况下,P3P 问题可能出现 1,2,3 或 4 个解.但是,若 3 个控制点和摄像机光心这 4 点共圆,则会出现无穷多组解.利用“蒙特卡洛”方法模拟出 P3P 问题分别出现 1,2,3,4 个解的概率为 0.9993,0.0007,0.0000,0.0000.结果论证了如下的事实,即在大多数情况下,P3P 问题有唯一解.
关键词: P3P 问题;多解现象;蒙特卡洛方法
中图法分类号: TP309 **文献标识码**: A

1 Introduction

The perspective-*n*-point problem, or the PnP problem, has been extensively studied over years in computer vision field due to its importance of camera or object pose determination. Among the PnP problem for different “n”, the P3P problem is the most fundamental one due to its wide applicability as well as its pivotal role-played for others. One of the main research directions for the P3P problem is the study on its multiple solutions. It is shown that the necessary and sufficient condition for the P3P problem to have an infinitely large number of solutions is the

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co-circularity of the three control points with the camera's optical center[1]. Fishler and Bolles[2] proved that the P3P problem has at most 4 positive solutions and this upper bound is also attainable via a concrete example. Wolfe, *et al*.[3] provided a geometric explanation of the distribution of the solutions for this problem, and showed that there are usually at most two solutions. Gao, *et al*.[4] gave a complete solution set of the P3P problem. Their results are purely algebraic, and seem more difficult to be instructive in real applications than directly solving a 4th degree polynomial as originally stated. In Ref.[5], it is shown that if the optical center lies within the danger cylinder, and additionally lies on any one of the three perpendicular planes going through the 3 altitudes of the control-point triangle, then the corresponding P3P problem must have 4 solutions. However, it seems difficult, if not impossible, to give such geometrical interpretations for all cases of multiple solutions of the P3P problem. Motivated by this, this manuscript is intended to have a study on the multiple solutions of the P3P problem from the probabilistic standpoint. In other words, based on a Monte Carlo Method, we would determine the probabilities of the P3P problem to have one, two, three, or four solutions.

2 Problem Statement

As we know, different randomness results in different probabilistic result. In order to correctly describe the P3P problem, the "correct randomness" must be used. Here by "correct randomness", we mean "the random sample process" involved in the Monte Carlo method must correctly represent the nature of the P3P problem.

2.1 The P3P definition and main constraints

It is defined as that given the relative spatial locations of 3 control points and given the angle to every pair of control points from the perspective center, find the distance of each of the control points from the perspective center.

As shown in Fig.1, *A*,*B*,*C* are three control points, *O* is the optical center, by the law of Cosines, we have the following familiar constraints

$$\begin{cases} x^2+y^2-2xy\cos\alpha=d_{AB}^2 \\ x^2+z^2-2xz\cos\gamma=d_{AC}^2 \\ z^2+y^2-2zy\cos\beta=d_{BC}^2 \end{cases} \tag{1}$$

where, d_{AB}=|AB|, d_{BC}=|BC|, d_{AC}=|AC|, and x=|OA|, y=|OB|, z=|OC| are the three distances to determine.

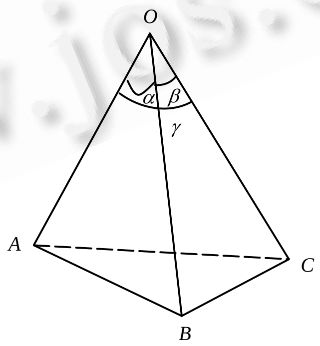


Fig.1 The geometry of the P3P problem

2.2 Logical randomness

In order to determine the probabilities of multiple solutions of the P3P problem by the Monte Carlo method, we think the following random sampling process is the logical one and should be adopted: Fix the optical center at the origin, then choose 3 control points at random in the 3D space, and determine the solutions of the corresponding P3P problem. Repeat the above process for N (very large) times, then the probability of having one, two, three, or four solutions is

$$p_i = \frac{N_i}{N}, \quad i = 1, 2, 3, 4.$$

Where, N_i is the number of the times having i solutions among the N trials and $\sum_{i=1}^{4} N_i = N$.

2.3 Transformation and boundedness

There is a practical problem for the above randomness to implement. This is because the involved 3D space is unbounded, as a result, it is difficult to implement the Monte Carlo method. Hence the first step we should take is to transform such an unbounded 3D space into a bounded one, which can be easily done by the following transformation

$$y = f(x) = \frac{x}{|x|+1} \tag{2}$$

The above transformation transforms an unbounded variable $x \in (-\infty, +\infty)$ into a bounded one $y \in (-1, +1)$. Applying the above transformation to each of the coordinates of the 3 control points, the unbounded problem can be solved as shown in Fig.2. Of course, if variable x has a uniform distribution, variable y will not be a uniform distribution. The two distribution densities have the following relationship.

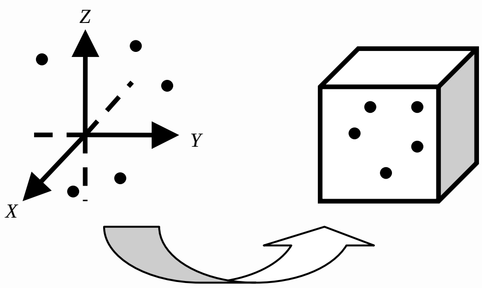


Fig.2 Space transformation

Assume x has a uniform distribution density $g(x)=\varepsilon$ with $x \in (-\infty, +\infty)$, then as shown in Ref.[6], variable y will have a distribution density like

$$g(y) = \frac{g(x)}{|f'(x)|} = \frac{\varepsilon}{(1-|y|)^2}, \; y \in (-1, +1) \tag{3}$$

3 Implementation of the Monte Carlo Method

The Monte Carlo Method is implemented as follows in our simulations:

Initialization: $N_1=N_2=N_3=N_4=0$

Repeat N (very large) times the following steps:

1. Choose 9 real numbers y_i $i=1,2,\ldots,9$ at random within $(-1,+1)$, by Eq.(3), determine their density values w_i $i=1,2,\ldots,9$ at these numbers.

2. From y_i $i=1,2,\ldots,9$, compute corresponding x_i $i=1,2,\ldots,9$ by Eq.(2).

3. Assume the optical center O at the origin, the 3 control points be (x_1,x_2,x_3), (x_4,x_5,x_6), (x_7,x_8,x_9), using the main constraints (1) to calculate the positive solutions. If this P3P problem has $i \in \{1,2,3,4\}$ positive solutions, then

$$N_i = N_i + \prod_{j=1}^{9} w_j .$$

Then the probability of having 1,2,3, or 4 solutions is

$$p_1 = \frac{N_1}{N_0},\quad p_2 = \frac{N_2}{N_0},\quad p_3 = \frac{N_3}{N_0},\quad p_4 = \frac{N_4}{N_0},\ N_0=N_1+N_2+N_3+N_4.$$

Remarks:

1. Denote

$$f_{AB}(x,y,z) := x^2 + y^2 - 2xy\cos\alpha - d_{AB}^2 ,$$
$$f_{AC}(x,y,z) := x^2 + z^2 - 2xz\cos\gamma - d_{AC}^2 ,$$
$$f_{BC}(x,y,z) := z^2 + y^2 - 2zy\cos\beta - d_{BC}^2 .$$

Due to numerical errors, a solution in our simulations is defined as

$$\mathrm{Max}\{|f_{AB}(x_i,y_i,z_i)|,|f_{AC}(x_i,y_i,z_i)|,|f_{BC}(x_i,y_i,z_i)|\}<\max\{a,b,c\}\times 10^{-3}.$$

Where, (x_i,y_i,z_i) is some computed solution.

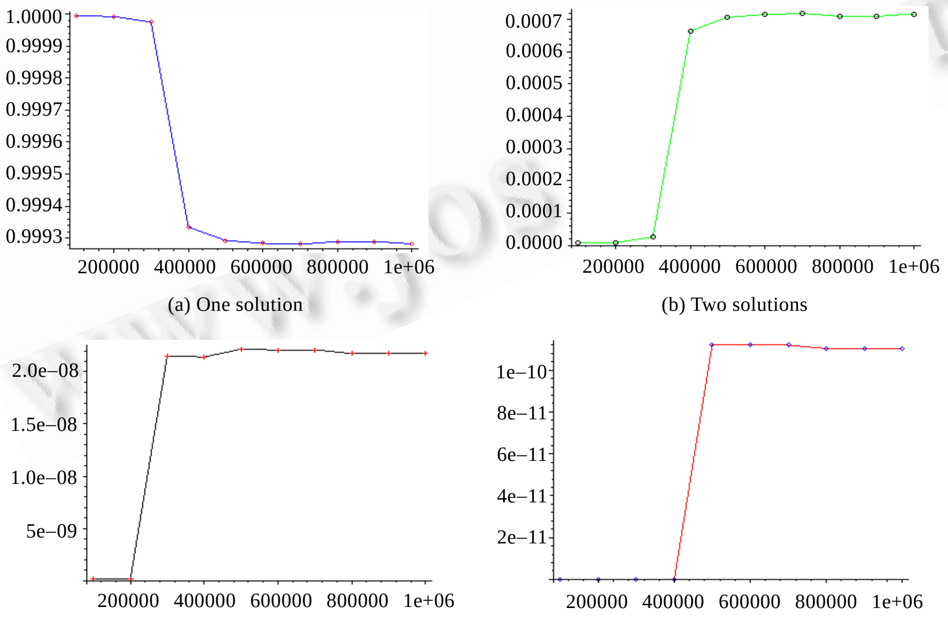
2. Due to numerical errors and the above criterion, although the P3P problem can have theoretically at most 4 solutions, in practice, cases where more than 4 solutions do exist. Due to quite limited number of such cases, they are ignored in our simulations.

4 Main Results

Based on 1 million trials, the results are shown in table 1 and Fig.3.

Table 1 The number of solutions and the corresponding probability

No. of solutions	1	2	3	4
Probability	0.9992825157	0.0007174625869	$0.2175806305\times 10^{-7}$	$0.1101550929\times 10^{-9}$



(c) Three solutions (d) Four solutions

Fig.3 Probability of having 1,2,3,or 4 solutions versus the number of trials

From these results, we see that the probability of having one or two solutions becomes stable after 300 000 trials. The probability of having 3 solutions is of the order of 10^{-7}, and that of having 4 solutions is of 10^{-9}. Compared with those of having 1 or 2 solutions, the probability of having 3 or 4 solutions can be safely considered as zero.

5 Conclusions and Discussions

By a Monte Carlo method, the probability of the P3P problem to have one, two, three, or four solutions are assessed. The results show that the probability of having one solution is dominant and as high as 99.928%. The probability of having 3 or four solutions is zero. These results are somehow instructive for a better understanding on the multiple-solution behavior of the P3P problem.

In fact, our simulations negate such an assumption, that is, by fixing 3 given control points, and let the optical center change, the corresponding P3P problem could have the same probabilistic results as listed in Section 4.

References:

[1] Su C, Xu YQ, Li H, Liu SQ, Li DG. Necessary and sufficient condition of positive root number of perspective-three-point problem. Chinese Journal of Computers, 1998,21(12):1084−1095 (in Chinese with English abstract).

[2] Fishler MA, Bolles RC. Random sample consensus: A paradigm for model fitting with applications to image analysis and automated cartography. Communications of ACM, 1981,24(6):381−395.

[3] Wolfe WJ, Mathis D, Sklair CW, Magee M. The perspective view of three points. IEEE Trans. on Pattern Analysis and Machine Intelligence, 1991,13(1):66−73.

[4] Gao XS, Hou XR, Tang JL, Cheng HF. Complete solution classification for the perspective-three-point problem. IEEE Trans. on Pattern Analysis and Machine Intelligence, 2003,25(8):930−943.

[5] Zhang CX, Hu ZY. A general sufficient condition of 4 positive solutions of the P3P problem. Journal of Computer Science & Technology, 2004,20(6):836−842.

[6] Papoulis A. Probability, Random Variables, and Stochastic Processes. 2nd ed., McGraw-Hill, Inc., 1985. 95−96.

附中文参考文献:

[1] 苏成,徐迎庆,李华,刘慎权,李冬果.判定 P3P 问题正解数目的充要条件.计算机学报,1998,21(12):1084−1095.

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