Perfect Approximation of Ellipsoid by Polynomials*

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Abstract: A simple method in the approximation of ellipsoid by bicubic polynomials is given in this paper, the error is approximately 273×10^{-6} for ellipse and 545×10^{-6} for ellipsoid.

Key words: approximation; ellipsoid; Bézier curves

As well known, Bézier curves and surfaces play a great role in the representation and design of free form curves and surfaces, but they could not denote circle, sphere and the like exactly. Hence in geometric modeling applications, the need of approximation of them arises when conic sections or rational curves, respectively, are not available or are not recommended.

On the other hand, CAD systems and the like always seem to offer some representation of cubic polynomials, no matter whether they use B-splines, Bézier or the plain canonical base for curves and surfaces.

Many authors have worked with the approximation of circle by Bézier polynomial[1~3]. In this paper, we consider the approximation of ellipsoid by bicubic polynomials, the error is approximately 273×10^{-6} for ellipse and 545×10^{-6} for ellipsoid.

1 The Approximation of Ellipse

An ellipse is defined as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a,b\in R. \tag{1.1}$$

We intend to use a piecewise cubic polynomial planar curve joining with continuous tangent directions in Bézier form to approximate the ellipse.

In order to approximate quadrant ellipse, we suppose that polynomial piece curve has the form:

$$p(t)=\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\sum_{i=0}^{3}P_iB_i^3(t),\quad t\in[0,1]. \tag{1.2}$$

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with

$$P_0=\begin{pmatrix}a\\0\end{pmatrix},\quad P_1=\begin{pmatrix}a\\ \frac{b}{3}\sigma\end{pmatrix},\quad P_2=\begin{pmatrix}\frac{a}{3}\sigma\\ b\end{pmatrix},\quad P_3=\begin{pmatrix}0\\b\end{pmatrix}\tag{1.3}$$

where

$$B_i^3(t)=\binom{3}{i}t^i(1-t)^{3-i}$$

is Bernstein basic function and $\delta\in R$ will be determined.

Simple calculation from (1.2) and (1.3) show that

$$x(t)=a(f_0(t)-\sigma g_1(t))=a(B_0^3(t)+B_1^3(t)+\frac{1}{3}\sigma B_2^3(t),$$

$$y(t)=b(f_1(t)+\sigma g_0(t))=b(B_2^3(t)+B_3^3(t)+\frac{1}{3}\sigma B_1^3(t)),\tag{1.4}$$

where $f_i(t)$, $g_i(t)$, i=0,1 are Hermite interpolant basic functions at point 0 and 1, so they are

$$f_0(t)=B_0^3(t)+B_1^3(t)=(1-t)^3+3t(1-t)^3,$$

$$f_1(t)=B_2^3(t)+B_3^3(t)=3t^2(1-t)+t^3,$$

$$g_0(t)=\frac{1}{3}B_1^3(t)=t(1-t)^2,\tag{1.5}$$

$$g_1(t)=-\frac{1}{3}B_2^3(t)=-t^2(1-t),$$

Thus they satisfy

$$f_i(j)=\delta_{ij},\quad f'_i(j)=0;$$

$$g'_i(j)=\delta_{ij},\quad g_i(j)=0,\qquad i,j=0,1.\tag{1.6}$$

Obviously Bézier curve $p(t)$ defined as (1.2) interpolates ellipse at point $(a,0)$ and $(0,b)$, and also has the same tangent direction at these two points.

We select δ so that

$$\frac{1}{a}x(1/2)=\frac{1}{b}y(1/2)=\frac{\sqrt{2}}{2}.\tag{1.7}$$

This means that the point $(x(1/2),y(1/2))$ corresponding to t=1/2 is on the ellipse. It is easy to get

$$\sigma=4(\sqrt{2}-1).\tag{1.8}$$

In order to estimate the error between ellipse (1.1) and Bézier curve (1.2), we introduce auxiliary function,

$$\rho(t)=\sqrt{\frac{x^2(t)}{a^2}+\frac{y^2(t)}{b^2(t)}}-1,\tag{1.9}$$

or

$$\varepsilon(t)=\frac{x^2(t)}{a^2}+\frac{y^2(t)}{b^2}-1,\tag{1.10}$$

for convenience.

Because $\varepsilon(t)$ is a polynomial of degree six, we write it in Bézier form:

$$\varepsilon(t)=\sum_{i=0}^{6}(b_i-1)B_i^6(t) \tag{1.11}$$

Using (1.4) and (1.6), it is easy to get $b_0=b_6=1$, $b_1=b_5=1$. From (1.5), we know

$$\begin{aligned} f_1(t)&=f_0(1-t),\\ g_1(t)&=-g_0(1-t),\\ g_1'(t)&=g_0'(1-t). \end{aligned} \tag{1.12}$$

These ensure that

$$\varepsilon(t)=\varepsilon(1-t). \tag{1.13}$$

Hence $b_2=b_4$ since the symmetry. $\varepsilon(t)$ can be rewritten as

$$\varepsilon(t)=c_1(B_2^6(t)+B_4^6(t))+c_2B_3^6(t) \tag{1.14}$$

with $c_1=b_2-1$ and $c_2=b_3-1$.

From (1.11), a simple calculation shows that

$$\varepsilon''(0)=30(b_2-2b_1+b_0)=30(b_2-1).$$

On the other hand, from (1.4) and (1.14), we also have

$$\varepsilon''(0)=2\sigma^2+4(\sigma-3)$$

where δ is defined as in (1.8). Therefore,

$$c_1=b_2-1=\frac{2}{15}(\sigma^2+2\sigma-6)=\frac{2}{15}(17-12\sqrt{2}). \tag{1.15}$$

Because the point $(x(1/2), y(1/2))$ is on the ellipse, we get the relation

$$3c_1+2c_2=0,$$

that is

$$c_2=-\frac{3}{2}c_1. \tag{1.16}$$

From (1.14), (1.15) and (1.16), we get

$$\varepsilon(t)=2(17-12\sqrt{2})t^2(1-t)^2(1-2t)^2. \tag{1.17}$$

It is easy to verify that the function $f(t)=t^2(1-t)^2(1-2t)^2$ on [0,1] at point $t=1/2-1/6\sqrt{3}$ obtains maximum, i.e.,

$$\max_{t\in[0,1]}t^2(1-t)^2(1-2t)^2=f\left(\frac{1}{2}-\frac{1}{6}\sqrt{3}\right)=\frac{1}{108}. \tag{1.18}$$

Finally, we have

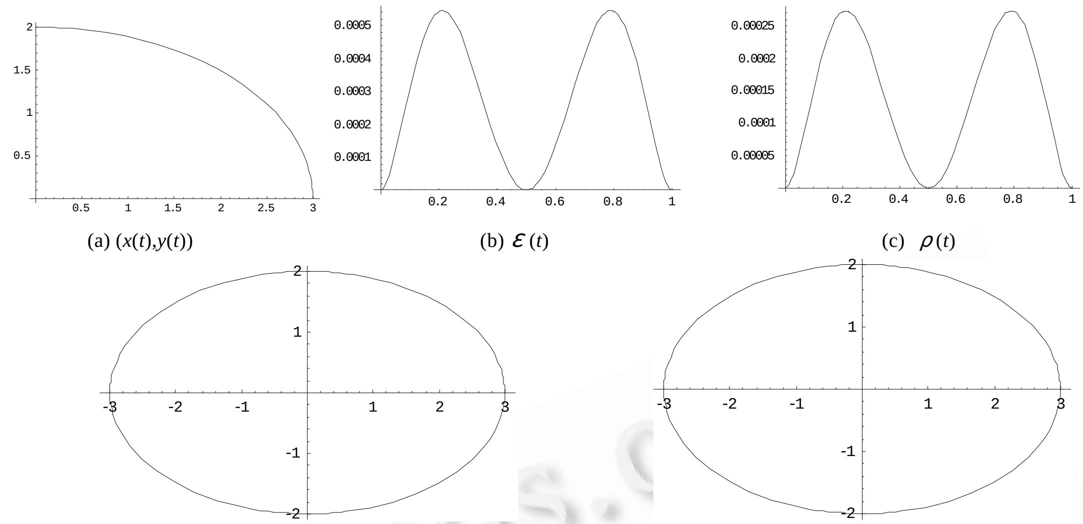
$$0\le\varepsilon(t)\le\frac{1}{54}(17-12\sqrt{2})\le 545\times10^{-6}, \tag{1.19}$$

and

$$0\le\rho(t)\le 273\times10^{-6}. \tag{1.20}$$

The error $\varepsilon(t)$ and $\rho(t)$ are the same with the result in Refs.[2,3] for the circle case ($a=b=1$).

We use Bézier curve $(x(t),y(t))$ defined in (1.2) to approximate a quadrant ellipse perfectly. We depict its picture in Fig.1(a), and we show the pictures of error functions $\varepsilon(t)$ and $\rho(t)$ in Figs.1 (b) and (c). It is easy to verify that the combination of $(x(t),y(t))$,$(x(t),-y(t))$,$(-x(t),y(t))$ and $(-x(t),-y(t))$ approximate the whole ellipse smoothly, see Fig.1(d).



(a) $(x(t),y(t))$ (b) $\varepsilon(t)$ (c) $\rho(t)$

(d) Smooth combination (e) The true ellipse

Fig.1 The approximation of an ellipse with a=3, b=2

2 The Approximation of Ellipsoid

The main purpose of this paper is to generalize the above method to surface case. An ellipsoid is defined as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ a,b,c\in R. \tag{2.1}$$

We define parametric polynomial surface $Q(s,t)$ using also Hermite interpolant as

$$\begin{cases} X(s,t)=ar(s)(f_0(t)-\sigma g_1(t)), \\ Y(s,t)=br(s)(f_1(t)-\sigma g_0(t)), \\ Z(s,t)=c(f_0(s)-\sigma g_1(s)), \end{cases} \tag{2.2}$$

where

$$r(s)=f_1(s)+\sigma g_0(s), \tag{2.3}$$

and f_i, g_i, i=0,1 and δ are defined as before.

We want to use surface $Q(s,t)$ to approximate octant ellipsoid. It is easy to check $Q(s,t)$ interpolates ellipsoid at points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$, and has the same tangent plane with ellipsoid at these three points.

Let

$$\rho^*(s,t)=\sqrt{\frac{X^2(s,t)}{a^2}+\frac{Y^2(s,t)}{b^2}+\frac{Z^2(s,t)}{c^2}}-1, \tag{2.4}$$

and

$$\varepsilon^*(s,t)=\frac{X^2(s,t)}{a^2}+\frac{Y^2(s,t)}{b^2}+\frac{Z^2(s,t)}{c^2}-1. \tag{2.5}$$

Note that from (1.4), (1.10), (2.2) and (2.3), we have

$$\frac{X^2(s,t)}{a^2}+\frac{Y^2(s,t)}{b^2}=r^2(s)(\varepsilon(t)+1), \tag{2.6}$$

and

$$\varepsilon(s)+1=\frac{Z^2(s,t)}{c^2}+r^2(s). \tag{2.7}$$

Hence

$$\varepsilon^*(s,t)=r^2(s)\varepsilon(t)+\varepsilon(s)=2(17-12\sqrt{2})\cdot \left(r^2(s)t^2(1-t)^2(1-2t)^2+s^2(1-s)^2(1-2s)^2\right) \tag{2.8}$$

Note that from (2.3) and (1.5), we know

$$r'(s)=2s(1-s)(3-\sigma)+\sigma(1-s)^2>0$$

and

$$r(s)\le r(1)=1\,, \tag{2.9}$$

therefore

$$r(s)\le 1.$$

From (2.8), (2.9) and (1.18), we get

$$\varepsilon^*(s,t)\le 109\times 10^{-5}, \tag{2.10}$$

and

$$\rho^*(s,t)\le 545\times 10^{-6}. \tag{2.11}$$

We can approximate the whole ellipsoid smoothly by changing the sign of $(X(s,t),Y(s,t),Z(s,t))$ respectively. For example, surface piece

$$Q_1(s,t)=(X(s,t),Y(s,t),-Z(s,t)) \tag{2.12}$$

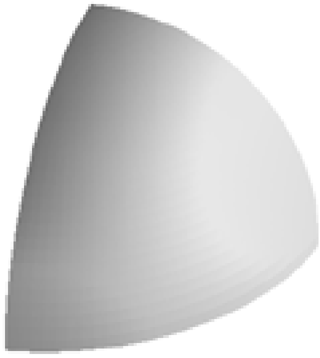
is continuous and has the same tangent plane with surface $Q(s,t)$ at common boundary curve $Q(1,t)$, and surface piece

$$Q_2(s,t)=(X(s,t),-Y(s,t),Z(s,t))$$

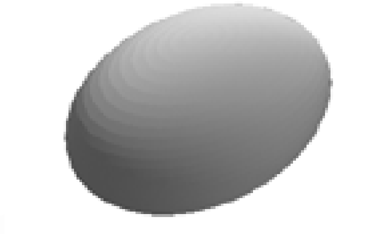
is also continuous and has the same tangent plane with surface piece $Q(s,t)$ at common boundary curve $Q(0,t)$. The effect of approximation is shown in Fig.2(b). In Fig.2(a), we depict the plot of $(X(s,t),Y(s,\mathrm{t}),Z(s,t))$, and we show the pictures of error functions in Fig.2(c) and Fig.2(d).

3 Conclusion

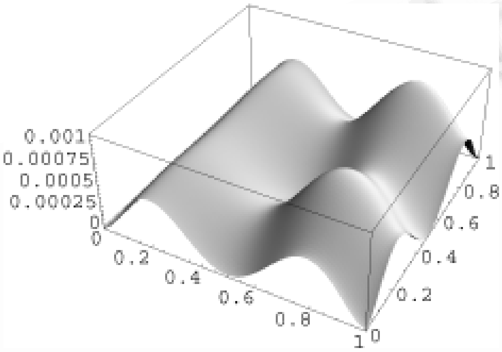
In this paper, the approximation of an octant ellipsoid surface by bicubic polynomials is considered and the whole ellipsoid surface is also approximated by piecewise bicubic polynomial patch with GC^1 continuity. The approximation error is about 0.0005. And, the method used in this paper can also be used to approximate an ellipsoid or a hyperboloid surface patch. We will discuss the more general situation in other paper.



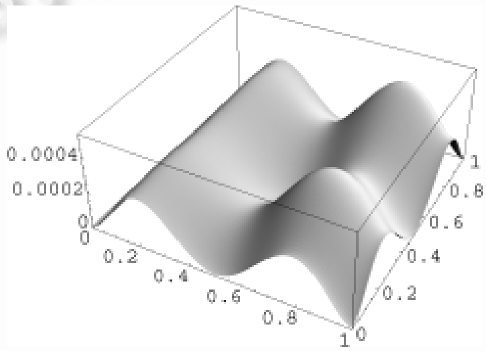
(a) $(X(s,t),Y(s,t),Z(s,t))$



(b) Smooth combination



(c) $\varepsilon^*(s,t)$



(d) $\rho^*(s,t)$

Fig.2 The approximation of an ellipsoid as a=3, b=2 and c=2

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椭球的高精度多项式逼近

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摘要: 给出了用双三次多项式逼近椭球的一种简明方法.逼近椭圆的误差为 273×10^{-6},逼近椭球的误差为 545×10^{-6}.

关键词: 逼近;椭球;Bézier 曲线

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