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Geometric Constraint Solving with Linkages*

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Abstract This paper introduces linkages as new drawing tool and shows that this tool is complete, i.e., all diagrams that can be described constructively can be drawn with linkages. This class includes the constraint problems with distance constraints only. As an application, the authors show that the simplest constrained graph which is beyond the scope of Owen and Hoffmann's popular triangle decomposition methods can be transformed to purely geometric constructive form. To solve the equations derived from linkage constructions, a geometric method which is based on dynamic locus generation is proposed.

Key words Geometric constraint solving, CAD, linkage, constrained graph, geometric method.

1 Introduction

Automated geometry diagram construction or geometric constraint solving (GCS) is the central topic in much of the current work of developing parametric CAD systems. It also has applications in mechanical engineering, chemical molecular modeling, linkage design, computer vision and computer aided instruction[1]. There are four main approaches to GCS: the graph analysis approach[2~5], the rule-based approach[6~8], the numerical computation approach[9,10], and the symbolic computation approach[11~13]. In practice, most people use a combination of these approaches to get the best result.

The graph analysis and the rule-based approaches are also called the geometric approach. In this approach, a pre-treatment is carried out to transform the constraint problem into a constructive form that is easy to draw. In most cases, this is equivalent to construct the diagram sequentially with ruler and compass. This can also be understood as drawing the diagram with geometric tools. But with ruler and compass, we can only draw a small portion of the diagrams. It is well known that using ruler and compass alone, we can describe diagrams whose equation systems are a sequence of triangularized equations of degree less than or equal to two. In Ref. [14], a new tool, conics, is added to enlarge the solving scope to diagrams that can be described by a sequence of triangularized equations of degree less than or equal to four. In this paper, we will introduce linkages as new tools and show that this tool is complete in certain sense, i.e., any general constructive diagram can be drawn with linkages sequentially. We also give an algorithm to find linkages in a constrained diagram. As an application, we proved that all well- or under- constrained problems containing point-to-point distance constraints only can be solved with linkages constructively.

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To solve the equations derived from linkage constructions, besides the often used numerical and symbolic computation methods, we introduce a geometric method which uses the linkages to generate loci and then finds the intersection of these loci by searching the points on the loci. This geometric method is based on dynamic generation of geometric locus which is widely used in dynamic geometric software[15~17].

Most of the results presented in this paper can be extended to 3D case.

As an application of the method introduced in this paper, we show that the simplest constrained graph which is beyond the scope of Owen and Hoffmann's triangle decomposition methods can be transformed to purely geometric constructive form if linkages are allowed as construction tools. The linkages used in the construction are three kinds of four-bar linkages.

The rest of this paper is organized as follows. Section 2 will show the drawing scope of using linkages as construction tools. Section 3 will present the geometric method for solving equations. In Section 4, we will show how to solve the simplest constrained graph.

2 Construction with Linkages

Most of the geometric approaches to GCS is to transform a constrained problem into constructive form with ruler and compass. We generalize this concept as follows. A geometric diagram can be drawn constructively or in constructive form if the geometric objects in it can be listed in an order

$$(O_1, O_2, \ldots, O_m),$$

such that each O_i can be determined by $(O_1, \ldots, O_{i-1})$ with a set of geometric constraints. Since all geometric objects can be treated as functions of points, we may assume without loss of generality that the geometric objects are points. The algebraic equations for a diagram in constructive form is naturally divided into blocks. In 2D case, the algebraic equations are as follows.

$$\begin{cases} f_{1,1}(u_1, \ldots, u_m, x_1, x_2) = 0 \\ f_{1,2}(u_2, \ldots, u_m, x_1, x_2) = 0 \\ f_{2,1}(u_1, \ldots, u_m, x_1, x_3, x_4) = 0 \\ f_{2,2}(u_2, \ldots, u_m, x_2, x_3, x_4) = 0 \\ \vdots \\ f_{t,2}(u_1, \ldots, u_m, x_1, \ldots, x_p) = 0 \end{cases} \tag{2.1}$$

Since the variables are introduced one by one or two by two, we may triangularize Eq. (2.1) easily, say, using resultant computation. Let the triangularized equations be

$$\begin{cases} t_1(u_1, \ldots, u_m, x_1) = 0 \\ t_2(u_2, \ldots, u_m, x_1, x_2) = 0 \\ \vdots \\ t_p(u_1, \ldots, u_m, x_1, \ldots, x_p) = 0 \end{cases} \tag{2.2}$$

It is well known that using ruler and compass alone, we can describe diagrams whose equation systems are of the form Eq. (2.2) and degree$(t_i) \leq 2$. In Ref. [14], a new tool, conics, is added to enlarge the scope to solve equation systems of form Eq. (2.2) and degree$(t_i) \leq 4$. A natural question is: can we add more tools such that the diagrams can be drawn with these tools covering all diagrams in constructive form. The answer is positive.

By a linkage, we mean a mechanism with one degree of freedom and consisting of links with fixed lengths and rotation joins. One example is the following four-bar linkage *ABCD-P* (Fig. 1). The locus of the four-bar linkage in the figure is generated as follows: with points A, B fixed and C rotating on a circle, point P will generate the locus.

A diagram can be drawn with linkages constructively if the points in the diagram can be listed in an order

$$(P_1, P_2, \ldots, P_m)$$

such that each point P_i is introduced by three basic constructions using the points already drawn $P_1, \ldots, P_{i-1}$.

(1) POINT(P): taking a free point P in the plane.

(2) ON(P,L): taking a semi-free point P on the locus L of a linkage.

(3) INTER(P,L_1,L_2): taking the intersection P of L_1 and L_2 which are the loci of two linkages.

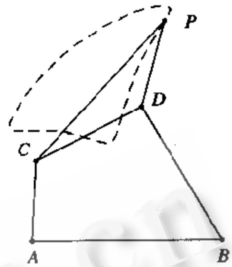


Fig. 1 The four-bar linkage and its locus

Theorem 2.1. A diagram is in constructive form iff it can be drawn with linkages.

Proof. It is readily seen that the locus of a linkage is an algebraic curve. Therefore, we need only to show that any diagram in constructive form can be constructed with linkages. This is valid because of a famous result of Kempe[18] which states that we may design a linkage to draw any given algebraic curve $f(x,y)=0$. In Ref. [18], we improved and implemented Kempe's result and showed that the complexity of the Kempe linkage is $O(n^4)$ where n is the degree of f. □

Since linkages could be very complicated, it seems that rule-based approaches are more appropriate to transform a constraint system into constructive form. For a rule-based system, like the global propagation method described in Ref. [6], we may add the following algorithm to find a linkage.

Algorithm 2.2. Suppose that we need to construct point P_0. We will find a linkage containing P_0. A point is said to be known if it has already been constructed.

S1 If there is a known point Q such that $|P_0Q|$ is known, then P_0 is on a circle. The algorithm terminates. Otherwise, let $S_0=\{P_0\}$ and go to S2.

S2 Let S_1 be the set of points such that $\forall P\in S_1, \exists Q\in S_0$, s.t. $|PQ|$ is known. If S_1 is an empty set, the algorithm terminates without finding a linkage.

S3 Let d be the number of distance constraints between pairs of points in $S_1 \cup S_0$ but not including pairs of two known points, n be the number of unknown points in $S_1 \cup S_0$.

S4 If $d=2n-1$, then the points in $S_1 \cup S_0$ consist of a linkage. The algorithm terminates.

S5 If $d>2n-1$, then there is an over-constrained sub-diagram. The algorithm terminates without finding a linkage. Otherwise, i.e., $d<2n-1$, let $S_0=S_1\cup S_0$ and go to S2.

Example 2.3. In Fig. 2, the lengths of the nine segments are known. Try to draw the diagram.

We may first draw triangle ABC. Next, we will determine point P. Since $|CP|$ is known, P is on a circle. Using the above algorithm we can find that point P is on a four-bar linkage $ABUV$-P. Then P is the intersection of a circle and the locus of the four-bar linkage $ABUV$-P.

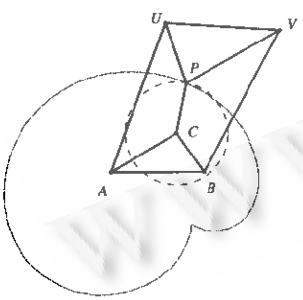


Fig. 2 Point P is the intersection of two loci

3 Evaluation of Construction Sequences of Linkages

Given a construction sequence $(C_1, C_2, \ldots, C_n)$, by introducing coordinates properly, we may obtain an equation system Eq. (2.1). Now we will show how to solve this equation system. Basically, we need to solve two algebraic equations.

$$\begin{cases} f(u_1,\ldots,u_m,x,y)=0 \\ g(u_1,\ldots,u_m,x,y)=0 \end{cases} \tag{3.1}$$

Please notice that in certain cases, the equations f and g are the equations of the loci for some linkages, which are not explicitly given.

If the linkage is complex, then it is difficult to find the equation of its locus. In this case, we may use the locus intersection method to find the solutions of Eq. (3.1). Suppose that we need to find the intersection of two loci L_1 and L_2. The locus intersection method has two main steps.

Generate Locus. Locus generation is a basic function of dynamic geometry[16]. It works as follows.

(1) Find a driving point which will move freely on a circle. In Fig. 1, a driving point could be C.

(2) Starting from this driving point, find a sequence of constructions with line and circle to construct the whole linkage.

(3) For each position of the driving point, we may compute the coordinates of the points in the linkage. In particular, the coordinates of the locus point.

(4) Repeating the preceding step, we have a set of coordinates of the locus point. We may use lines or Bezier curves to connect two neighboring points to form a continuous locus.

Find Intersection. After the two loci L_1 and L_2 are generated, we search them to find two points $P_1 \in L_1$ and $P_2 \in L_2$ such that $|P_1P_2|$ has minimal value. Notice that there might be more than one solution.

In practice, this method is quite efficient, because to generate the locus we need only to solve linear and quadratic equations which have closed form solutions[17].

We first use an example to illustrate the method presented in the preceding section.

Example 3.1. As shown in Fig. 3, the lengths of the nine segments are known. Try to draw the diagram.

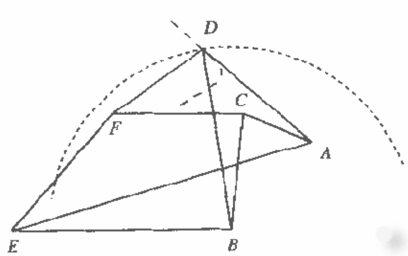


Fig. 3　A constrained problem with six points

We may first draw points E, B. Now point D is the intersection of a circle and the locus of a linkage $EBFCA$. Let F be the driving point. The construction sequence for the linkage is as follows:

ON(F,CIR(E,$|EF|$))
INTER(C,CIR(B,$|BC|$),CIR(F,$|FC|$))
INTER(A,CIR(E,$|EA|$),CIR(C,$|CA|$))
INTER(D,CIR(A,$|AD|$),CIR(F,$|FD|$))

where CIR(B,$|BC|$) represents the circle with center B and radius $|BC|$. With the above construction sequence, we may generate the locus for point D when point F moves on CIR(E,$|EF|$). Fig. 3 is actually generated in this way by a software named Geometry Expert[17].

Both Examples 2.3 and 3.1 contain point-to-point distance only. It is not difficult to check that they are the two smallest possible constraint problems of this kind that can not be solved by ruler and compass construction. We will show that all constraint problems of this kind can be solved by linkages constructively.

Theorem 3.2. All well- or under- constrained problems containing point-to-point distance constraints only can be solved with linkages constructively.

Proof.　We need only consider well-constrained problems since under-constrained problems may become well-constrained problems by adding appropriate number of point-to-point distance constraints. We assume that the problem contains n points. Then it must have $2n-3$ constraints. Let us assume that $|AB|$ is known. We first draw A,B. Let C be a point such that AC is known. We will construct point C. Since AC is known, it is already on a circle. If BC is also known, we may construct C as the intersection of two circles. Repeat the above process until we cannot go further. Let S be the set of points constructed in this way, T be the set of the remaining points, and $k=|S|$, $t=|T|$. Then $k+t=n$.

There must be points $P\in T$ and $Q\in S$ such that $|PQ|$ is known. We will construct P which is already on a circle since $|PQ|$ is known. The number of constraints not used in S is $2(n-k)$. Since $|PQ|$ is also used, we have $2(n-k)-1$ constraints left. For the point set T to form a linkage, we need $2t-1=2(n-k)-1$ constraints. Then by Algorithm 2.2, T forms a linkage, and P is the intersection of a circle and the locus of this linkage. The remaining points can be treated similarly. □

4 A Smallest Triconnected Constrained Graph

Hoffmann and Owen's triangle decomposition method is one of the most popular methods of GCS. Constrained graphs that can be solved by these methods are non-triconnnected graphs[3,4]. As it is pointed out in Ref. [2], the simplest constrained graph that cannot be solved with these methods is the following graph. The vertices of the graph could be a point or a line. The edges represent geometric constraints:

Pair of vertices	Geometric constraint represented by the edge
Point/Point	Distance between two points
Line/Line	Angle formed by the two lines
Point/Line	Coincidence or distance from point to line

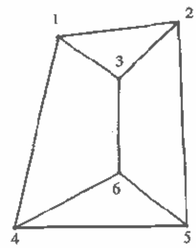


Fig. 4 Smallest triconnected graph

Since each vertex of the constrained graph in Fig. 4 could be a point or a line, we may introduce a notation to represent the graph: $(V_1V_2V_3, V_4V_5V_6)$ where V_i could be P or L. If $V_i=P$, then the i-th position in Fig. 4 is a point. If $V_i=L$, then the i-th position in Fig. 4 is a line. With this notation, Fig. 4 represents 13 types of constrained graphs.

Theorem 4.1. All the 13 problems can be solved with linkages constructively.

Table 1 gives the information on how to solve the 13 problems.

In Table 1, P/L means the type of point-line constraint; Type means whether the problem is well-, over- or under-constrained; R/C means whether the problem can be drawn with ruler and compass; Locus one (two) means the most complicated loci or linkages needed in the construction.

Some of the cases have been considered. For instance, cases 8 and 8' are solved in Ref. [3] with Göbner basis method. We will show that all of the thirteen cases can be solved constructively if linkages are allowed as drawing tools.

Of the thirteen cases, five use linkages. Case 1 is Example 2.3. Case 2 is similar to case 1. Cases 3', 6' and 8' need two new types of linkages.

An *ll*-four-bar linkage consists of two fixed lines u, v and a triangle PAB with fixed shape such that $A\in u$ and $B\in v$. The locus is generated by point P (Fig. 5(a)). This linkage is denoted by $(uvAB, P)$.

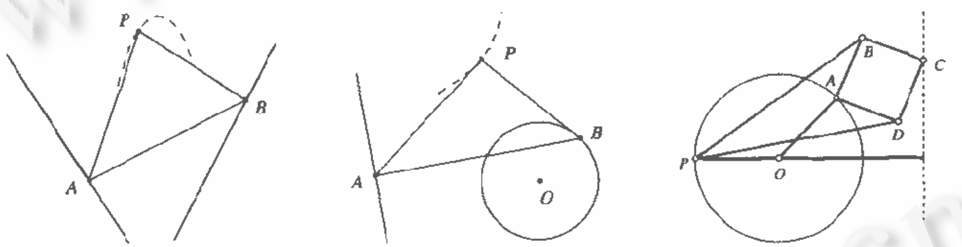
An lc-four-bar linkage consists of a fixed line l, a fixed circle c and a triangle PAB with fixed shape such that $A\in l$ and $B\in c$. The locus is generated by point P (Fig. 5(b)). This linkage is denoted by $(ucAB, P)$.

By the definition of linkages, a point cannot move on a line. This problem can be solved with the famous Peaucelier linkage which may generate a straight line (Fig. 5(c)).

Figure 6(a) is the geometric diagram for case 3'. We may first draw uvP. Since distance(B,u) and distance (C,v) are known, B and C move on two lines and we have an ll-linkage $(uvBC, A)$. Now A is the intersection of circle CIR$(P, |PA|)$ and the locus of the ll-four-bar linkage $(uvBC, A)$.

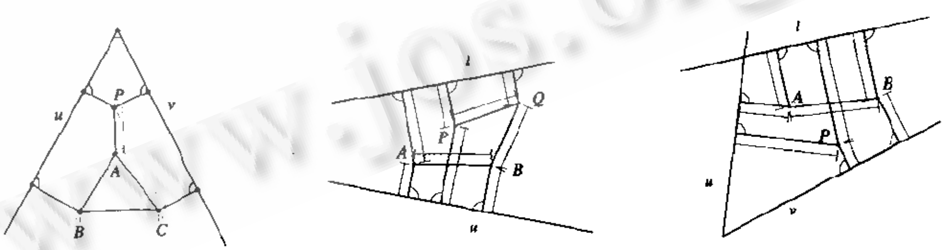
Table 1 Thirteen triconnected constraint graphs

	Problem	P/L	Type	R/C	Locus one	Locus two
1	(PPP,PPP)	Any	Well	No	circle	four-bar
2	(PPP,PPL)	Any	Well	No	line	four-bar
3	(PPP,PLL)	Coincidence	Well	Yes	circle	circle
3'	(PPP,PLL)	Distance	Well	No	line	ll-four-bar
4	(PPP,LLL)		Over			
5	(PPL,PPL)	Any	Well	Yes	circle	circle
6	(PPL,LPP)	Coincidence	Under	Yes	line	circle
6'	(PPL,LPP)	Distance	Well	No	line	lc-four-bar
7	(PPL,PLL)	Any	Well	Yes	line	circle
8	(PPL,LLP)	Coincidence	Under	Yes	line	circle
8'	(PPL,LLP)	Distance	Well	No	line	ll-four-bar
9	(PPL,LLL)		Over			
10	(PLL,PLL)		Over			
11	(PLL,LLP)	Any	Well	Yes	line	circle
12	(PLL,LLL)		Over			
13	(LLL,LLL)		Over			



(a) (b) (c)

Fig. 5 Two new four-bar linkages



(a) (b) (c)

Fig. 6 Three constaint problems which need four-bar linkages

Cases 6' and 8' need special explanation. Figure 6(c) is the geometry diagram for case 8'. We first draw the diagram Puv. Next, we will draw line l. Since distance(l,P) is known, l is tangent to a circle. Then we may generate the locus of l. Note that ABl is a rigid body and points A and B move on two lines. Then we may use an ll-four-bar linkage to simulate the movement of the rigid body ABl, and to generate the locus of l. The position of l can be determined as the intersection of the two loci of lines. Case 6' can be treated similarly.

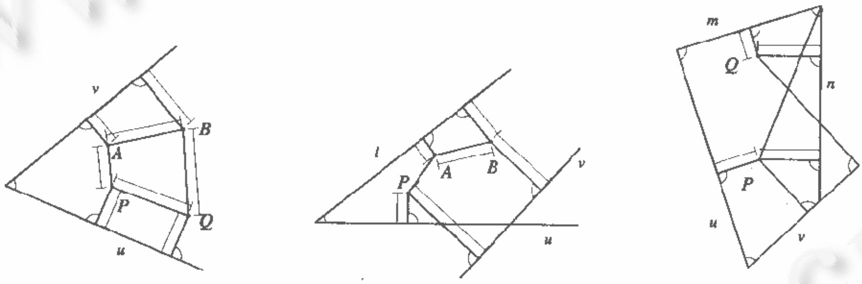
Cases 4, 9, 10, 12, 13 are over-constrained, because there are conflicting constraints. However, if these constraints in them are compatible, all of the constrained systems are under-constrained system and can be drawn with ruler and compass easily.

Cases 5, 7 and 11 can be drawn with ruler and compass. They can be solved with the Global Propagation method in Ref. [6].

• To solve case 5 (Fig. 7(a)), we first draw PQu. Since ABv and PQu are rigid bodies, we know the angle formed by lines AB and PQ. Now the problem is transformed into the following one: "draw a quadrilateral if we know the lengths of its four sides and the angle formed by a pair of opposite sides", which has been solved in Ref. [6].

• To solve case 6 (Fig. 7(b)), we first draw uvP. Next, we will draw point A. Since $|PA|$ is known, A is on a circle c. Similar to case 5, we know the angle between lines AB and u. Since distance(B,v) is known, B is on a line l_1. Since $|AB|$ and the direction of line AB are known and B moves on line l_1, by transformation $B \to A$, A must move on another line l_2. A is the intersection of c and l_2. For details about this kind of transformation, see Ref. [6].

• To solve case 11 (Fig. 7(c)), we first draw uvP. Next, we will draw line n. Since distance $|nP|$ is known, n is tangent to a circle. Since $\angle(n,m)$ and $\angle(m,u)$ are known, we know the direction of n. Now n is a line with known direction and tangent to a known circle, and thus can be determined.



(a) (b) (c)

Fig. 7 Three problems which can be solved with ruler and compass

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用连杆机构几何约束求解

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摘要　引入连杆机构作为新的工具,且证明这是完备的,也就是说,所有能构造性描述的图形能被连杆机构作出,这一类包括了所有只含距离约束的约束问题.作为一个应用,说明了超出Owen和Hoffmann的三角分解方法之外的最简单的约束图能被转化为纯几何构造形式.为了求解起源于连杆构造的方程,提出了一种基于动态轨迹生成的几何方法.

关键词　几何约束求解,连杆机构,三连通,计算机辅助设计,关系图.

中图法分类号　TP391